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Abstract

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UNITARY REPRESENTATIONS IN THE SPACE G/Γ, WHERE G IS THE GROUP OF REAL MATRICES OF ORDER n, AND Γ IS THE SUBGROUP OF INTEGRAL MATRICES

The present work is devoted to the study of the unitary representation in the space $X = G/\Gamma$, where G is the group of real matrices of order n, and Γ is the subgroup of integral matrices. The principal tool is the method of horocycles. Let us recall the definitions of horocycles and horocyclic subgroups.

Let G be a real semisimple Lie group, and let $g(t)$ be some one-parameter subgroup of the group G. The set $Z \subset G$, consisting of all z for which

$$\lim_{t\to\infty} g(-t)zg(t) = 1,$$

is called the **horocyclic subgroup associated with the subgroup** $g(t)$. Let X be a homogeneous space of the group G. **Horocycles** in X are the orbits of horocyclic groups. A horocycle is called **compact** if the set of points of which it consists is compact.

If G is the group of real matrices of order n, then there exist as many nonconjugate horocyclic subgroups as there are representations of n in the form of a sum of positive summands $n = k_1 + k_2 + \cdots + k_s$ (representations differing in order are considered distinct). The corresponding horocyclic subgroups have the form

$$\begin{pmatrix} E_{k_1} & * & \cdots & * \\ 0 & E_{k_2} & \cdots & * \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & E_{k_s} \end{pmatrix}, \tag{1}$$

where below the "diagonal" there stand zeros, and above it arbitrary numbers; E_{k_i} denotes the identity matrix of order k_i.

We denote the subgroup (1) by $Z_{k_1,\dots,k_s}$. The following theorem holds, describing the structure of all compact horocycles in the space $X = G/\Gamma$.

Theorem 1. *Every compact horocycle in X is the image of a set $Z_{k_1,\dots,k_s}g_0$ under the natural mapping of G onto X, where g_0 denotes an arbitrary fixed ele-*

ment of the group G. The horocycles $Z_{k_1,\dots,k_s}g_0$ and $Z_{k'_1,\dots,k'_{s'}}g'_0$ are transformed into one another by motions from G if and only if $k_1 = k'_1, \dots, k_s = k'_s, s = s'$.

Let us briefly outline the proof of this theorem. From the compactness of the quotient space $Z_{k_1,\dots,k_s}/\Gamma \cap Z_{k_1,\dots,k_s}$ it follows that the image of the set $Z_{k_1,\dots,k_s}g_0$ is a compact horocycle in X. It is also evident that the images of the sets $Z_{k_1,\dots,k_s}g_0$ and $Z_{k'_1,\dots,k'_{s'}}g'_0$ in X are transformed into one another by motions from G if and only if the groups $Z_{k_1,\dots,k_s}$ and $Z_{k'_1,\dots,k'_{s'}}$ are conjugate in G,

and, consequently, if and only if $s = s'$, $k_1 = k'_1, \dots, k_s = k'_s$. More difficult is the proof of the fact that every compact horosphere in X is the image of the set $Z_{k_1,\dots,k_s}g_0$. For this one must first prove that if Z is a horospherical subgroup in G such that the quotient space $Z/\Gamma \cap Z$ is compact, then Z has the form $h^{-1}Z_{k_1,\dots,k_s}h$, where h is some matrix with rational entries. Then, using uniqueness of factorization in the ring of integers, it is shown that Z has the form $\gamma^{-1}Z_{k_1,\dots,k_s}\gamma$, where $\gamma \in \Gamma$.

We now pass to the consideration of functions on X. Denote by $H^0_{k_1,\dots,k_s}$ the collection of all functions in $L_2(X)$ whose integrals over all compact horospheres of the form $Z_{k_1,\dots,k_s}g_0$ are equal to zero. It is easy to see that the space $H^0_{k_1,\dots,k_s}$ is invariant with respect to the translation operators T_g. Next denote by $H'_{k_1,\dots,k_s}$ the intersection of all spaces $H^0_{k'_1,\dots,k'_s}$ corresponding to the groups $Z_{k'_1,\dots,k'_s}$ that contain $Z_{k_1,\dots,k_s}$ as a proper subgroup. It is readily checked that $H^0_{k_1,\dots,k_s} \subset H'_{k_1,\dots,k_s}$. We shall further denote by $H_{k_1,\dots,k_s}$ the quotient space $H'_{k_1,\dots,k_s}/H^0_{k_1,\dots,k_s}$. In $H_{k_1,\dots,k_s}$ the action of the operators T_g is defined in a natural way. Finally, denote by H^0 the collection of all functions $f(x)$ in $L_2(X)$ whose integrals over all compact horospheres are equal to zero.

Theorem 2. The space $L_2(X)$ is isomorphic to the sum of the spaces $H_{k_1,\dots,k_s}$ and H^0

$$L_2(X) \cong \sum_{k_1+\dots+k_s=n} H_{k_1,\dots,k_s} + H^0. \tag{2}$$

In our paper (1) it was shown that H^0 decomposes into a sum of a countable number of irreducible unitary representations of the group G. In the present paper we shall deal with the decomposition into irreducible representations of the spaces $H_{k_1,\dots,k_s}$.

Apart from its immediate interest, the significance of this problem also lies in the fact that it leads to a number of remarkable analytic functions closely connected with such functions as the classical Riemann zeta-function and its generalizations. It is not excluded that a deep development of the theory of these and analogous functions will shed light also on the unresolved questions in the theory of the classical Riemann zeta-function. We shall now give the definition of these functions for $H_{k_1,\dots,k_s}$. The general definition for arbitrary regular groups will be given later.

Consider the set Ω_1 of all compact horospheres of maximal dimension. Since a motion carries a compact horosphere again into a compact one, it is natural to define an action of the group G on Ω_1.

It follows from Theorem 1 that the group G acts transitively on Ω_1. To each function $f(x) \in L_2(X)$ we assign its integral over the compact horospheres from Ω_1, and denote it by $\check{f}(\omega)$. Let $\mathcal{L}$ denote the collection of functions $\check{f}(\omega)$ on Ω_1 that are integrals of functions $f(x) \in L_2(X)$. $\mathcal{L}$, evidently, is isomorphic to $H_{1,1,\dots,1}$.

Denote by $L_2(\Omega)$ the collection of all square-summable functions on Ω_1. The decomposition of $L_2(\Omega)$ into irreducible representations is carried out, as is known, in the following way. Denote by A the connected component of the group of left translations of the space Ω, i.e. transformations $\omega \mapsto a\omega$ commuting with motions.

The group A is naturally identified with the group of diagonal matrices with positive entries on the diagonal. Denote by

by $a^\chi = \exp(\chi \ln a)$, where $\ln a$ denotes the canonical mapping of A into its Lie algebra. Put

$$\Phi(\chi, \omega) = \int_A \varphi(a\omega)\, a^{-\chi} j^{1/2}(a)\, da, \tag{3}$$

where $j(a)$ is the Jacobian of the transformation $\omega \to a\omega$, which depends only on a.

Formula (3), where χ is purely imaginary, gives a decomposition of $L_2(\Omega)$ into irreducible representations; moreover, two representations with "numbers" χ and $\tilde{\chi}$ are equivalent if and only if $\tilde{\chi}(a) \equiv \chi(a^\sigma)$ for all a, where a^σ denotes a with permuted eigenvalues; σ is a permutation.

It can be shown that $\mathcal{L}$ decomposes into the sum of two spaces $\mathcal{L}'$ and $\mathcal{L}_1$, where $\mathcal{L}'$ consists only of constants, and $\mathcal{L}_1 \subset L_2(\Omega)$. It turns out that in $\mathcal{L}_1$, unlike in $L_2(\Omega)$, each irreducible representation occurs once. Therefore the $\Phi(\chi, \omega)$ for functions from $\mathcal{L}_1$, at points corresponding to equivalent irreducible representations, turn out to be related. Put

$$\Phi(\sigma\chi, \omega) = \xi_\sigma(\chi)\Phi(\chi, \omega). \tag{4}$$

The functions $\xi_\sigma(\chi)$ are of principal interest. They are meromorphic in the whole complex space and satisfy the functional equation

$$\xi_{\sigma_1\sigma_2}(\chi) = \xi_{\sigma_1}(\sigma_2\chi)\xi_{\sigma_2}(\chi). \tag{5}$$

The latter is equivalent to the fact that each representation in $\mathcal{L}_1$ occurs once. Using the functional equation (4), one can show that

$$\xi_\sigma(\chi) = \prod_\alpha \theta((\chi, \alpha)),$$

where $\theta(s) = B\left(\frac{1}{2}, \frac{s}{2}\right)\zeta(s)\zeta^{-1}(s+1)$, $\zeta(s)$ is the classical Riemann zeta-function, and α runs through a certain set of roots, which for each σ consists of all negative roots α for which $\sigma\alpha$ is a positive root; the brackets denote the Cartan scalar product; $\sigma\alpha$ is the root obtained from α by the substitution σ.

The spaces $H_{k_1,\dots,k_s}$ are studied analogously. It turns out that all of them are the sum of two spaces $\hat{\mathcal{L}}_{k_1,\dots,k_s}$ and $\mathcal{L}'_{k_1,\dots,k_s}$, the first of which is embedded in $L_2(\Omega_{k_1,\dots,k_s})$* and decomposes into the same irreducible representations as $L_2(\Omega_{k_1,\dots,k_s})$, but contains them with smaller multiplicity. The latter circumstance leads to functional equations for the arising analogues of the functions $\xi_\sigma(\chi)$. $\mathcal{L}'_{k_1,\dots,k_s}$ consists of representations corresponding to special points of the functions $\xi_\sigma(\chi)$ and their analogues.

Let us briefly outline the proof for $H_{1,\dots,1}$. (We shall denote the scalar product of two functions $\check{f}_1, \check{f}_2 \in \mathcal{L}$ by $[\check{f}_1, \check{f}_2]$.) We shall also agree, for any two functions $\varphi_1(\omega)$ and $\varphi_2(\omega)$ on Ω, for which the integral $\int \varphi_1\overline{\varphi}_2\, d\omega$ converges absolutely, to denote its value by (φ_1, φ_2).

It is easy to show that for every finite continuous function $\varphi(\omega)$ on Ω the expression $(\check{f}, \varphi)$ has meaning for all $\check{f} \in \mathcal{L}$ and is a linear functional on $\mathcal{L}$. According to Riesz' s theorem, in $\mathcal{L}$ there exists such

* $\Omega_{k_1,\dots,k_s}$ denotes the space of horospheres associated with the group $z_{k_1,\dots,k_s}$.

a vector M_φ such that $(f, \varphi) = [\check{f}, M_\varphi]$ for all $\check{f} \in \mathcal{L}$. The operator M plays a fundamental role in further investigations. For M there exists a comparatively simple explicit expression, which we shall now give. First let us agree on the following notation: $\Delta = \Gamma \cap Z$; Δ' is the normalizer of Δ in Γ; $Z_g = Z \cap g^{-1}Zg$, where $g \in G$; D_g is the set of left cosets of the adjacent group with respect to the subgroup Z_g. Now put

$$M_{g_0}(\varphi(\omega)) = \int_{Z_{g_0}} \varphi^*(g_0 z g)\, dz, \tag{6}$$

where $\varphi^*(g)$ denotes the function on G corresponding to the function $\varphi(\omega)$ on Ω.

The following formula holds

$$M_{\varphi} = \sum M_{\gamma}, \tag{7}$$

where γ ranges over the set of all representatives of the double cosets $\Delta'\gamma\Delta$ of the group Γ with respect to the subgroups Δ' and Δ.

The subsequent arguments are based on consideration of the form $(M\varphi_1, \varphi_2)$ and on the study of the Mellin transform of functions of the form $M\varphi$, where φ denotes a continuous finite function.

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References

1. I. M. Gel' fand, I. I. Pyatetskii-Shapiro, DAN, **147**, No. 1 (1962).

Note: Figure translations are in progress. See original paper for figures.