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MATHEMATICS

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DYADIC BICOMPACTA AND CONTINUOUS MAPPINGS OF ORDERED BICOMPACTA

(Presented by Academician P. S. Aleksandrov on 23 XI 1961)

This note briefly sets forth the proof of one conjecture of P. S. Aleksandrov* (Theorem 4) and gives some other results concerning bicompacta possessing the Suslin property and at the same time being continuous images of ordered bicompacta.

By a continuous image of an ordered bicompactum we mean any Hausdorff space that can be obtained as a continuous image of at least one ordered bicompactum (in its natural topology). A bicompactum X has the Suslin property if every system of nonempty open pairwise disjoint subsets of X is at most countable. E. Špilerain proved (^{1}) that every dyadic bicompactum has the Suslin property.

Definition 1. A continuous mapping $f : K \to X$ of a bicompactum K onto a space X is called **quasi-open** if the image $f(U)$ of every open nonempty set $U \subseteq K$ has interior points, i.e. if the interior $\operatorname{Int} f(U) \neq 0$.

Definition 2. A continuous mapping $f : K \to X$ of an ordered bicompactum $(K, \leq)$ onto X is called **light** if for every point $x \in X$ its full preimage $f^{-1}(x)$ contains no closed interval from K having more than one point.

Lemma 1. *If X is a continuous image of an ordered bicompactum, then there exist an ordered bicompactum K and a continuous quasi-open and light mapping $f : K \to X$ of the bicompactum K onto X.*

Proof of Lemma 1. Let K be an ordered bicompactum and let $f : K \to X$ be a continuous mapping onto X; furthermore, let $\mathfrak{F}$ be the system of all closed sets $K_\alpha \subseteq K$ for which $f(K_\alpha) = X$. Partially order the system $\mathfrak{F}$ by putting $K_\alpha \leq K_\beta$ if $K_\alpha \supseteq K_\beta$. It is not hard to prove that every linearly ordered subsystem $\mathfrak{F}' \subseteq (\mathfrak{F}, \leq)$ has an upper bound. Hence, applying Zorn' s lemma, we make sure that in $(\mathfrak{F}, \leq)$ there exists at least one maximal element K_1. It is easy to see that the restriction $f_1 = f|K_1$ of the mapping f to K_1 is a quasi-open mapping.

To obtain lightness of the mapping as well, one must further contract to a point each closed interval $[a, b]$ in K_1 whose image $f_1([a, b])$ is only one point in X. In this way we obtain a new ordered bicompactum K_2 and a mapping $f_2 : K_2 \to X$ onto X, which is at the same time both light and quasi-open.

Theorem 1. *Let K be an ordered bicompactum and let $f : K \to X$ be a continuous, quasi-open, and light mapping onto X. If X has the Suslin property, then K satisfies the first axiom of countability.*

Proof of Theorem 1. The set $Z \subseteq X$ of isolated points of X is at most countable, but it is easy to show (taking into account the lightness of the mapping f) that then the set $f^{-1}(Z)$ is also at most countable.

For isolated points $t \in K$, the assertion of the theorem is obvious. Suppose now that t is a limit point, for example, for the set $(., t)_K$ of all points $s \in K$, $s < t$. Then t is the exact least upper bound of a monotonically increasing transfinite sequence of points

$$s_0 < s_1 < \cdots < s_\xi < \cdots, \qquad \xi < \omega_\alpha, \tag{1}$$

where each of the intervals $(s_\xi, s_{\xi+1})$ is nonempty. We must prove that $\omega_\alpha = \omega_0$. Otherwise we could construct (taking into account the assumptions on the function f) a subsequence ξ_ζ, $\zeta < \omega_\alpha$, such that

$$V_\xi = \operatorname{Int} f(U_{\xi(\zeta)}) \setminus [f(U_{\xi(\zeta+1)})] \neq 0, \tag{2}$$

where by $U_{\xi(\zeta)}$ we denote the open interval $(s_{\xi(\zeta)}, t) \subseteq K$, and by $[A]$ the closure of the set A. (Here, without loss of generality, one may assume that $(s_0, t) \cap f^{-1}(Z) = 0$.) But the set $\{V_\zeta,\ \zeta < \omega_\alpha\}$ is an uncountable disjoint system of nonempty open sets in X, contrary to the assumption.

Theorem 2. *Let X be a continuous image of an ordered bicompactum. In order that the bicompactum X have the Suslin property, it is necessary and sufficient that every open set $V \subseteq X$ be an F_σ-set.*

The sufficiency is proved without difficulty for every bicompactum X. The proof of necessity is based on the following lemma:

Lemma 2. *Let K be an ordered bicompactum and $f : K \to X$ a quasi-open mapping onto a bicompactum X possessing the Suslin property. Then for every open set $V \subseteq X$, the set $f^{-1}(V)$ is the union of $\leq \aleph_0$ disjoint open intervals.*

Proof of Lemma 2. By transfinite induction one can construct a system $\mathcal{G}$ of open disjoint nonempty sets G_α, such that the set

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* The conjecture was stated at the Fourth All-Union Mathematical Congress in Leningrad, 3 VII–12 VII 1961.

$$\bigcup_{G_\alpha \in \mathcal{G}} [G_\alpha] = H \subseteq V \tag{3}$$

is dense in V. Since X has the Suslin property, $\mathcal{G}$ is a countable system. The open set $f^{-1}(V)$ is the union of a disjoint system of open intervals U_λ belonging to the set K, each U_λ being a maximal interval contained in $f^{-1}(V)$. For each $G \in \mathcal{G}$ the bicompact set $f^{-1}([G])$, obviously, is contained in a finite number of intervals U_λ. Since $\mathcal{G}$ is a countable system, in view of condition (3) we conclude that $f^{-1}(H)$ has a nonempty intersection with at most a countable number of intervals U_λ. But the mapping f is quasi-open, and therefore $f^{-1}(H)$ intersects every U_λ. Hence it follows that the set of all U_λ is at most countable.

The proof of necessity in Theorem 2 is now obtained as follows. Let K be an ordered bicompactum and $X = f(K)$. By Lemma 1 and Theorem 1 we may assume that K satisfies the first axiom of countability and that f is quasi-open. But then every open interval of K is an F_σ-set. From Lemma 2 it now follows that, for every open set $V \subseteq X$, $f^{-1}(V)$ is an F_σ-set. Hence it follows that V is also an F_σ-set.

Theorem 3. *If a bicompactum X has the Suslin property and is a continuous image of an ordered bicompactum, then X satisfies the first axiom of countability.*

It suffices to prove that each point $x \in X$ is a G_δ-set, but this is an immediate consequence of Theorem 2.

Theorem 4. *A dyadic bicompactum X is a continuous image of an ordered bicompactum if and only if it is metrizable.*

The sufficiency is obvious, because the Cantor discontinuum is an ordered dyadic bicompactum.

The necessity of metrizability follows from the fact that X, by the aforementioned theorem of Shpilrain, has Suslin' s property and, by Theorem 3, satisfies the first axiom of countability. But this property entails the metrizability of the bicompactum X on the basis of a theorem of A. S. Esenin-Vol' pin (^{2}), who proved that every dyadic bicompactum satisfying the first axiom of countability is metrizable.

In an entirely different way one can strengthen Theorem 1 and obtain the following theorem:

Theorem 5. *Let K be an ordered bicompactum and $f : K \to X$ a continuous, quasi-open, and light mapping onto X. The bicompactum X has Suslin's property if and only if K has Suslin' s property.*

As a consequence of Theorem 5 and of certain results of the authors of the present note (^{3}), the following theorem is obtained:

Theorem 6. *The following three assertions are equivalent to one another:*

S_1. *Every ordered bicompactum K having Suslin' s property contains a countable everywhere dense set.*

S_2. *Every bicompactum X having Suslin' s property and being a continuous image of an ordered bicompactum contains a countable everywhere dense set.*

S_3. *Every continuum X having Suslin' s property and being a continuous image of an ordered continuum is metrizable.*

Remark. Whether assertion S_1 is true is a well-known unsolved problem of M. Ya. Suslin.

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Note: Figure translations are in progress. See original paper for figures.

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