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**Abstract**

**Full Text**

**I. P. EGOROV**

**MAXIMALLY MOBILE EINSTEIN SPACES OF NON-CONSTANT CURVATURE**

*(Presented by Academician A. N. Kolmogorov, 21 III 1962)*

1. As is known, the order  $r$  of the groups of motions  $\mathfrak{G}_r$  of  $n$ -dimensional Riemannian spaces  $V_n$  is not greater than  $n(n+1)/2$ . The spaces of constant curvature, and only they, possess maximal mobility. In the theory of motions of spaces  $V_n$  the following theorem, due to G. Fubini <sup>(1)</sup>, is also known: there do not exist Riemannian spaces  $V_n$  admitting complete groups of motions  $\mathfrak{G}_r$  of order  $r = n(n+1)/2 - 1$ . In the paper <sup>(4)</sup> we showed that the Fubini theorem described is inaccurate in the distribution of the orders of complete groups of motions. In the present paper a theorem is proved on the nonexistence of non-Einstein spaces  $V_n$  (properly Riemannian or pseudo-Riemannian) admitting complete groups of motions  $\mathfrak{G}_r$ , whose order  $r$  satisfies the inequalities

$$n(n-1)/2 + 1 < r < n(n+1)/2;$$

further (the second theorem), if  $V_n$  admits  $\mathfrak{G}_r$  of order  $r = n(n-1)/2 + 2$ , then it is Einstein (i.e., the Ricci tensor is proportional to the fundamental tensor); if  $r > n(n-1)/2 + 2$ , then the space  $V_n$  is also Einstein and necessarily is a space of constant curvature. Thus, there do not exist Einstein spaces of nonconstant curvature with groups  $\mathfrak{G}_r$  of order  $r > n(n-1)/2 + 2$ . G. Vrânceanu noted <sup>(2)</sup> that the estimate following from this for the maximal order  $r = n(n-1)/2 + 2$  in the case of properly Riemannian Einstein spaces is exact only for  $n = 4$ . He proved that properly Riemannian Einstein spaces  $V_n$  of nonconstant curvature possess groups of motions  $\mathfrak{G}_r$  of order  $r \leq (n-1)(n-2)/2 + 4$  ( $n \geq 7$ ). Further, Wakakuwa <sup>(3)</sup>, relying on Montgomery's results on subgroups of the orthogonal group, showed that, under the same conditions with respect to the metric ( $ds^2 > 0$ ), this order is  $r \leq (n-1)(n-2)/2 + 3$ , starting from some  $n$  ( $n > 249$ ); the bound  $r = (n-1)(n-2)/2 + 3$  is evidently exact.

In the general case, when the metric of the space  $V_n$  is also allowed to be indefinite, the problem of determining the order of the complete groups of motions of maximally mobile Einstein spaces of nonconstant curvature had not been solved. In the present paper a solution of this problem is given, and the exactness of the established order  $r$  of the groups of motions of the spaces under consideration is shown.

2. Let the Einstein space  $V_n$  of nonconstant curvature with fundamental tensor  $g_{ij}(x)$  be referred to an orthonormal frame  $\lambda_a^i$  ( $a = 1, 2, \dots, n$ ) so that

$$ds^2 = e_1\omega_1^2 + e_2\omega_2^2 + \dots + e_n\omega_n^2, \quad (1)$$

where  $\omega_a = \omega_{ai}(x) dx^i$  are Pfaff forms,  $e_a = \pm 1$ . If  $V_n$  admits an infinitesimal motion defined by the vector field  $\xi^\sigma(x^1, \dots, x^n)$ , then the quantities  $a_p = \xi^i \lambda_{pi}$ ,  $b_{pq} = \lambda_p^i \lambda_q^j \xi_{i,j}$  (here and in what follows covariant differentiation is taken with respect to  $g_{ij}(x)$ ) satisfy, as follows—

known, following mixed system of differential equations:

$$\begin{aligned} \frac{\partial a_p}{\partial s_q} &= b_{pq} + \sum_r e_r a_r v_{prq}, & b_{pq} + b_{qp} &= 0, \\ \frac{\partial b_{pq}}{\partial s_r} &= \sum_s e_s (a_s R_{srqp} + b_{ps} v_{qsr} + b_{qs} v_{spr}), \end{aligned} \quad (2)$$

where  $\partial/\partial s^q$  denotes the derivative in the direction along the vector field  $\lambda_q^i$ ;  $v_{pqr}$  are the Ricci coefficients, and  $R_{abcd}$  are the coordinates of the curvature tensor referred to a nonholonomic frame. The first integrability conditions of this system of equations with respect to  $a_p$  and  $b_{pq}$  reduce to the form

$$\sum_s (b_{sa} R_{sbcd} + b_{sb} R_{ascd} + b_{sc} R_{absd} + b_{sd} R_{abcs}) \equiv 0,$$

where the equality is written up to additive terms containing  $a_p$ , and moreover  $R_{sbcd} = e_s R_{sbcd}$ . If  $b_{\alpha_p \alpha_q}$  ( $p < q$ ) are taken as the principal quantities, then the preceding relations can be written in the form:

$$(abcd) = \sum b_{\alpha_p \alpha_q} T_{abcd}^{\alpha_p \alpha_q} = 0, \quad (3)$$

$$\begin{aligned} T_{abcd}^{\alpha_p \alpha_q} &= \delta_a^{\alpha_p} R_{\bar{\alpha}q bcd} + \delta_b^{\alpha_p} R_{a\bar{\alpha}qcd} + \delta_c^{\alpha_p} R_{ab\bar{\alpha}qd} + \delta_d^{\alpha_p} R_{abc\bar{\alpha}q} - \\ &- \delta_a^{\alpha_q} R_{\bar{\alpha}p bcd} - \delta_b^{\alpha_q} R_{a\bar{\alpha}p cd} - \delta_c^{\alpha_q} R_{ab\bar{\alpha}p d} - \delta_d^{\alpha_q} R_{abc\bar{\alpha}p}. \end{aligned} \quad (4)$$

Let us now consider, at some point of the space  $V_n$ , the Weyl conformal curvature tensor  $C^a_{bcd}$ ; if the indices  $a, b, c, d$  are distinct, then in the orthoframe  $\lambda_a^i$  we shall have

$$C^a_{bab} = R^a_{bab} - \frac{Re_b}{n+1}, \quad C^a_{bad} = R^a_{bad}, \quad C^a_{bcd} = R^a_{bcd}. \quad (5)$$

This tensor for the (Einstein) spaces under consideration is not equal to zero; otherwise  $V_n$  is a space of constant curvature.

3. The following interesting proposition holds:

**Theorem.** *The maximal order  $r$  of complete groups of motions  $G_r$  of Einstein spaces  $V_n$  of nonconstant curvature is exactly equal to*

$$(n-1)(n-2)/2 + 5.$$

For the proof of the theorem we shall consider the integrability conditions (3) of the system of equations (2), using a partial reduction of some coordinates of the curvature tensor (in a nonholonomic frame) to zero (2).

We next construct the forms

$$\Phi_e^a = R^a_{bed}\omega_b\omega_d,$$

where  $\omega_b$  are the Pfaff forms occurring in the expression of the line element of the space. In view of the nonvanishing of the Weyl tensor, as formulas (5) show, one of the forms  $\Phi_a^a \neq ds^2 \pmod{\omega_a}$ , or one of the forms  $\Phi_b^a$  ( $a \neq b$ ), is not identically zero. In what follows it is possible to restrict ourselves to the first case only; if all the forms  $\Phi_a^a$  are comparable with  $ds^2$  modulo  $\omega_a$ , then one of  $\Phi_b^a$  ( $a \neq b$ ) is different from zero and, applying a pseudo-orthogonal or orthogonal transformation of the forms  $\omega_a, \omega_b$  (according as  $e_a \neq e_b$  or  $e_a = e_b$ ), we shall obtain, by changing the numbering of the vectors of the orthoframe, the form  $\Phi_1^1 \neq 0$ . Now we shall subject the forms  $\omega_2, \omega_3, \dots, \omega_n$  also to such a transformation, preserving  $ds^2$ , under which  $\Phi_1^1$  assumes the canonical form (6).

Let  $\Phi_1^1$  be of simple type and the roots  $\lambda_i$  of the characteristic equation be real (at least two roots must be distinct); then, by virtue of (2), we arrive at the presence of  $2n-5$  ( $n \geq 7$ ) relations in the conditions (3).

If among the roots  $\lambda_k$  there are complex ones  $\alpha_k + i\beta_k$  ( $i = \sqrt{-1}$ ), then, combining them into complex-conjugate pairs, one can, by a real transformation of the forms  $\omega_2, \dots, \omega_n$  that does not change the form of  $ds^2$ , reduce the matrix of the form  $\Phi_1^1$  to diagonal block form with second-order submatrices

$$\begin{pmatrix} \alpha_k - \beta_k & \sqrt{2}\beta_k \\ \sqrt{2}\beta_k & -(\alpha_k + \beta_k) \end{pmatrix}$$

and of first order. When  $\Phi_1^1$  has characteristic  $[1, \bar{1}(1, 1, \dots, 1)]$ , the matrix of the system of equations  $(|a_2|a_3), (|a_2|a_l), (|a_3|a_l)$  ( $l = 4, \dots, n$ ) with respect to  $b_{a_2a_3}, b_{a_2a_l}, b_{a_3a_l}$  is nondegenerate and

$$r \leq (n-1)(n-2)/2 + 4.$$

It is not difficult also to verify the nondegeneracy of this matrix for  $[(1, 1, \dots, 1, 1)(\bar{1}, \bar{1}, \dots, \bar{1})]$  and other combinations of real and complex roots with simple elementary divisors.

Let us now consider the second extreme case, when the matrix of the form  $\Phi_1^1$  consists of a single Jordan block; if  $n$  is odd ( $n = 2k + 1$ ), then the system (12|a), (13|a),  $\alpha = 4, \dots, 2k + 1$ , with respect to  $b_{\beta 2k}, b_{\beta 2k+1}$ ,  $\beta = 3, \dots, 2k - 1$ , leads us to a matrix of order  $2n - 6$ , whose determinant is equal in absolute value to two. If  $n$  is even ( $n = 2k$ ) and  $\Phi_1^1$  has characteristic  $[2k - 1]$ , then the equations (12|a), (13|a),  $\alpha = 4, \dots, n$ , with respect to  $b_{\beta n-1}, b_{\beta n}, b_{4, n-2}$  give a nondegenerate matrix. Suppose now that  $\Phi_1^1$  has characteristic  $[2k + 1, \overline{2k + 1}]$  or  $[2k, \overline{2k}]$  for  $n = 4k + 3$  or  $n = 4k + 1$ , respectively. The matrices of these forms are obtained from the matrices

$$\begin{pmatrix} 4\alpha & 0 & 4 & -2\sqrt{2} & 4\beta & 0 \\ 0 & 4\alpha & 0 & 2\sqrt{2} & 0 & -4\beta \\ 4 & 0 & 4(\alpha - \beta) & 4\sqrt{2}\beta & 0 & -4 \\ -2\sqrt{2} & 2\sqrt{2} & 4\sqrt{2}\beta & -4(\alpha + \beta) & -2\sqrt{2} & 2\sqrt{2} \\ 4\beta & 0 & 0 & -2\sqrt{2} & -4\alpha & 0 \\ 0 & -4\beta & -4 & 2\sqrt{2} & 0 & -4\alpha \end{pmatrix}, \quad (6)$$

$$\begin{pmatrix} 4\alpha + 2 & 2 & 4\beta - 2 & -2 \\ 2 & 4\alpha - 2 & 2 & -4\beta - 2 \\ 4\beta - 2 & 2 & -4\alpha - 2 & 2 \\ -2 & -4\beta - 2 & 2 & -4\alpha + 2 \end{pmatrix}$$

by applying  $(k - 1)$  times a double cyclic bordering with the elements

$$[4\alpha]0, 2, 0, \dots, 0, 2, \varepsilon, [0], -\varepsilon, -2, 0, \dots, 0 - 2, 0[-4\alpha],$$

where for the first and second borderings each time  $\varepsilon = 0$  or  $\varepsilon = 4\beta$ , respectively (the elements enclosed in square brackets are corner elements). Further, we verify that the equations (12|a), (13|a) ( $\alpha = 4, \dots, n$ ) are solvable with respect to  $b_{2\beta}, b_{3\beta}$  ( $\beta = 4, \dots, n$ ). If the matrix  $\Phi_1^1$  splits into a pair of elementary blocks, then the matrix of system (3) splits into three matrices, which makes it possible to establish the corresponding solvability in the other cases as well.

To prove that the obtained estimate of the maximal order  $r$  of the group of motions  $\mathfrak{G}_r$  is sharp, we shall consider the Riemannian space defined by the linear element

$$ds^2 = dx^{12} + dx^{22} - dx^{32} - dx^{42} + \left[ \sum_{ij=12, 13, 24, 34} (x^i dx^j - x^j dx^i)^2 \right] + e_5 dx^{52} + e_6 dx^{62} + \dots + e_{ndx}^n, \quad e_a = \pm 1 \quad (a = 5, 6, \dots, n). \quad (7)$$

This is an Einstein space of nonconstant curvature, and it has the simplest structure of a non-simple form  $\Phi_1^1$ ; it has zero scalar curvature, decomposes into the direct product of a space  $V_4$  with the first 4 variables and the flat  $S_{n-4}$  of the remaining variables. The space (7) is of the first class and its metric is induced on a second-order hypersurface of the flat  $S_{n+1}$ . The space  $V_n$  with line element (7) is also of interest from another point of view: for  $n \geq 4$  it is an  $(n-2)$ -fold projective non-Kagan space (the direct product of a 2-fold projective non-subprojective space  $V_4$  and a projective space  $S_{n-4}$ ). This space admits a group of motions  $G_r$  of order

$$r = (n-1)(n-2)/2 + 5.$$

We have 8 operators corresponding to the first 4 variables;  $(n-4)(n-3)/2$  operators corresponding to the variables  $x^5, x^6, \dots, x^n$ , and  $2(n-4)$  operators connecting all variables:

$$\begin{aligned} X_1 &= p_2 - p_3, & X_2 &= p_1 + p_4, \\ X_3 &= p_2 + (x^4 - x^1)(x^2 + x^3)(p_1 + p_4)(x^4 - x^1)^2(p_2 - p_3), \\ X_4 &= p_4 - [(x^2 + x^3)^2(p_1 + p_4) + (x^4 - x^1)(x^2 + x^3)(p_2 + p_3)], \\ X_5 &= (x^2 + x^3)(p_1 + p_4) + (x^4 - x^1)(p_2 - p_3), \\ X_6 &= x^4 p_1 + x^3 p_2 + x^2 p_3 + x^1 p_4, \\ X_7 &= (x^2 + x^3)(p_4 - p_1) + (x^1 + x^4)(p_2 - p_3), \\ X_8 &= (x^3 - x^2)(p_4 + p_1) + (x^1 - x^4)(p_2 + p_3), \\ X_{0a} &= p_a, & X_{ab} &= e_a x^a p_b - e_b x^b p_a \quad (\text{no summation over } a, b), \\ Y_{1a} &= e_a x^a (p_3 - p_2) + (x^2 + x^3)p_a, & Y_{2a} &= e_a x^a (p_4 + p_1) + (x^4 - x^1)p_a, \\ & & a, b &= 5, 6, \dots, n. \end{aligned} \tag{8}$$

The curvature tensor of the  $V_n$  under consideration has a very special structure:

$$R_{\alpha\beta\gamma\delta} = \varepsilon \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta} \quad (\varepsilon = \pm 1), \tag{9}$$

where  $\varepsilon_{\alpha\beta}$  is necessarily a simple bivector; this bivector is covariantly constant and nilpotent, i.e.

$$\varepsilon_{\alpha\beta, \gamma} = 0; \tag{10}$$

$$\varepsilon_{\alpha}{}^{\sigma} \varepsilon_{\sigma\beta} = 0. \tag{11}$$

It is not difficult to verify that spaces satisfying the last three requirements are Einstein spaces, the order of whose full groups is exactly

$$(n-1)(n-2)/2 + 5.$$

4. Let us note the consequences that follow directly from the theorem proved:

- 1) There do not exist Riemannian spaces  $V_n$  admitting motion groups  $G_r$  of order  $r$ , if for  $n > 6$

$$(n-1)(n-2)/2 + 5 < r < n(n-1)/2$$

(as an additional consideration shows, this result is also valid for  $n \geq 4$ ).

- 2) The maximally mobile Riemannian spaces  $V_n$  of the third lacunarity are Einstein spaces with motion groups  $G_r$  of order  $r = (n-1)(n-2)/2 + 5$  or  $r = (n-1)(n-2)/2 + 4$  <sup>(5)</sup>.
- 3) The maximal order of the motion groups  $G_r$  of nonconformally Euclidean spaces is exactly  $(n-1)(n-2)/2 + 5$ . The value  $(n-1)(n-2)/2 + 5$  must be replaced by  $(n-1)(n-2)/2 + 3$  if we restrict ourselves to a positive-definite metric.

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*Note: Figure translations are in progress. See original paper for figures.*

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