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# MATHEMATICS

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**Abstract**

**Full Text**

**MATHEMATICS**

**V. K. IVANOV**

**INTEGRAL EQUATIONS OF THE FIRST  
KIND AND AN APPROXIMATE SOLUTION  
OF THE INVERSE PROBLEM OF POTENTIAL  
THEORY**

*(Presented by Academician S. L. Sobolev on 23 X 1961)*

1. Let  $X$  and  $Y$  be Banach spaces; let  $A$  be a continuous (linearity is not assumed) operator mapping  $X$  into  $Y$ . We shall consider the equation

$$Ax_0 = y_0, \tag{1}$$

where  $x_0 \in X$  is unknown and  $y_0 \in Y$  is a given element, in the absence of continuous dependence of  $x_0$  on  $y_0$ , i.e., when the inverse operator  $A^{-1}$  is not continuous.

Equation (1) is an abstract analogue of integral equations of the first kind, which have a number of applications in function theory and mathematical physics. Such equations were studied by M. M. Lavrent'ev, who, under certain additional assumptions, gave approximate methods for their solution <sup>(1)</sup>. We give a method for solving equation (1), proceeding from different considerations and under different conditions. In our approach, finding an approximate solution of equation (1) is reduced to finding the minimum of a function of a finite number of variables in a prescribed domain. A similar idea is applied in <sup>(2)</sup> to the problem of continuation of a harmonic function. As an application, we consider the construction of a convergent process which, under certain restrictions, gives an approximate solution of the integral equation of the inverse problem of potential theory.

2. We shall assume that, for the given  $y_0$ , equation (1) has a solution  $x_0$  in a class  $M$  of elements of  $X$ , where  $M$  satisfies the following conditions:
  - 1)  $M$  is a set of uniqueness of solutions of equation (1), i.e., if  $x' \in M$ ,  $x'' \in M$ , and  $Ax' = Ax''$ , then  $x' = x''$ .
  - 2)  $M$  is compact in itself.
  - 3) There exists an increasing chain of finite-dimensional closed subsets of the set  $M$ :

$$M_1 \subset M_2 \subset \dots \subset M_n \subset \dots; \quad M_n \in M, \quad (2)$$

such that  $x_0$  is a limit point of the sum  $\sum_{n=1}^{\infty} M_n$ . Obviously, all  $M_n$  are compact in themselves.

3. Denote  $AM_n = N_n$  ( $n = 1, 2, \dots$ ). From the condition in item 2 it follows that

$$\lim_{n \rightarrow \infty} \rho(x_0, M_n) = 0.$$

Then, by the continuity of  $A$ , we shall have

$$\lim_{n \rightarrow \infty} \rho(y_0, N_n) = 0. \quad (3)$$

Let  $x$  range over  $M_n$ . Then  $F_n(x) = \|Ax - y_0\|$  is a continuous numerical function defined on  $M_n$ . Owing to the compactness and closedness of  $M_n$ , this function attains at some point  $x_n$  of  $M_n$  its minimum, equal to  $\rho(y_0, N_n)$ . Thus, in  $M_n$  there is a point  $x_n$  such that

$$\rho(y_0, N_n) = \|Ax_n - y_0\|. \quad (4)$$

From (3) and (4) there follows the convergence

$$Ax_n \rightarrow y_0. \quad (5)$$

Put  $AM = N$ . From the continuity of  $A$  and conditions 1) and 2) of item 2 it follows that the inverse operator  $A^{-1}$ , taking  $N$  into  $M$ , exists and is continuous on  $N$  (3). Hence, on the basis of (5),

$$x_n \rightarrow x_0. \quad (6)$$

The elements  $x_n$  may be taken as approximate values of  $x_0$ . Finding  $x_n$  reduces to finding the minimum of the function  $F_n(x)$ , defined on the finite-dimensional compact set  $M_n$ , which can be done effectively (see (4,5)). As was shown in (1), in the presence of an error in specifying  $y_0$ , the best possible approximation to  $x_0$  is obtained for some definite  $n$ . Finding such an optimal  $n$  can be carried out by a method analogous to that used in (1).

4. As an example, let us give the construction of a convergent process for the approximate solution of the inverse problem of potential. For simplicity we shall restrict ourselves to consideration of the plane problem.

Let there be situated inside the unit circle with center at the origin a domain  $D$ , star-shaped with respect to the origin and filled with matter of density one. On the circle  $r = 1$  the logarithmic potential  $y(\varphi)$  of the domain  $D$  is known. The problem is to find  $D$ . Suppose that the polar equation of the boundary  $C$  of the domain  $D$  is  $r = x(\varphi)$ . For the potential  $y(\varphi)$  we have

$$y(\varphi) = \int_0^{2\pi} \int_0^{x(\alpha)} \ln \frac{1}{\sqrt{1 - 2r \cos(\varphi - \alpha) + r^2}} r dr d\alpha.$$

Taking the inner integral, we arrive at the equation

$$\int_0^{2\pi} K[\varphi, \alpha, x(\alpha)] d\alpha = y(\varphi). \quad (7)$$

As  $X$  we take the set of all continuous functions of period  $2\pi$  with the metric  $C$ , and as  $Y$  the space  $L_2[0, 2\pi]$ . The operator  $A$  is defined by (7); the problem reduces to finding  $x(\varphi)$  from (7). We shall assume that the solution  $x_0(\varphi)$  exists in the class  $M$  of functions from  $X$

$$x(\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\varphi + b_k \sin k\varphi),$$

whose Fourier coefficients satisfy the relations

$$\sum_{k=1}^{\infty} (|a_k| + |b_k|) \leq \frac{a_0}{2} \leq \frac{B_1}{2}; \quad (8)$$

$$\sum_{k=1}^{\infty} k(|a_k| + |b_k|) \leq B_2, \quad (9)$$

where  $0 < B_1 < 1$ ,  $0 < B_2$  are given constants.

From (8) follows the star-shapedness of the body bounded by the curve  $r = x(\varphi)$ ; therefore, by the theorem of P. S. Novikov <sup>(5)</sup>,  $M$  is a set of uniqueness

for solutions of equation (7). Further, from (8) and (9) there follow the estimates

$$0 \leq x(\varphi) \leq B_1 < 1, \quad |x'(\varphi)| \leq B_2,$$

whence the compactness of  $M$  follows.

As  $M_n$  we take the set of all trigonometric polynomials belonging to  $M$  of degree not exceeding  $n$ .

$$P_n(\varphi) = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos k\varphi + b_k \sin k\varphi).$$

The problem of finding an approximating polynomial  $x_n(\varphi)$  from  $M_n$  reduces to finding the minimum of the function

$$F_n(a_0, a_1, \dots, a_n, b_1, \dots, b_n) = \int_0^{2\pi} \left[ \int_0^{2\pi} K(\varphi, \alpha, P_n(\alpha)) d\alpha - y_0(\varphi) \right]^2 d\varphi$$

in the  $2n + 1$  variables  $a_0, a_1, \dots, b_n$  inside the  $(2n + 1)$ -dimensional convex polyhedron defined by the inequalities

$$\sum_{k=1}^n (|a_k| + |b_k|) \leq \frac{a_0}{2} \leq \frac{B_1}{2},$$

$$\sum_{k=1}^n k(|a_k| + |b_k|) \leq B_2.$$

On the basis of the results of Sec. 3, the sequence of minimizing trigonometric polynomials  $x_n(\varphi)$  converges uniformly to the required solution.

In an analogous way one can give a solution of the inverse problem of potential also in space, considering functions  $x(\theta, \varphi)$  and  $y(\theta, \varphi)$  defined on the sphere. The potential  $y(\theta, \varphi)$  may be given only on part of the sphere or on part of some plane. Other schemes are also possible; for example, approximation (in the plane case) of the required curve by a polygonal line.

Ural State University  
named after A. M. Gorky

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*

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