# Mathematics

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## Abstract

## Full Text

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# Various Types of Convergence of Cubature and Quadrature Formulas

The error functional of a cubature formula

$$(l, f) = \int f \, d\Omega - \sum C_k f(X^{(k)}) \tag{1}$$

can be studied in various topologies.

S. M. Nikol' skii ([1]), the author ([2]), and others considered the question of the maximum of $(l, f)$ on the unit sphere of the Banach space $B$:

$$\max_{\|f\|_B = 1} (l, f) = d(l). \tag{2}$$

From this point of view, the convergence of cubature formulas depending on the number $N$ reduces to the study of the convergence of the numbers $d(l^{(N)})$ for the corresponding functionals $(l^{(N)}, f)$. Another, equally common, approach to the convergence of cubature and quadrature formulas is convergence with respect to order of proximity. In the present note we shall dwell on this latter point of view.

In formula (1), instead of a numerical function $f$, we may consider an abstract function with values in some Banach space $X$ or topological space $\tau$

$$f(x) \in X \quad \text{or} \quad f(x) \in \tau. \tag{3}$$

The function $f$ itself, mapping $R_n$ into $X$ or $\tau$, will then be an element of some Banach space $B$ or, in the more general case, of a topological space $T$:

$$f(R_n \to X) \in B \quad \text{or} \quad f(R_n \to \tau) \in T. \tag{4}$$

The error functional $(l, f)$ is thereby transformed into an error operator, mapping $B$ or $T$ into $X$ or $\tau$, since both $\int f \, d\Omega$ and $\sum C_k f(x^{(k)})$ will be elements of $X$ or $\tau$. The convergence of cubature formulas to zero will be characterized by the tendency to zero of the operators $l^{(N)}$.

As $X$ it is convenient to consider the space of countable vectors:

$$a \in X, \qquad a = (a_1, a_2, \dots, a_n, \dots) \tag{5}$$

with some Banach norm, for example $l_2$ or $m$.

An abstract function $f$ with values in $X$ or $\tau$ will in this case take values

$$f = (f_1, f_2, \dots, f_n, \dots). \tag{6}$$

Convergence with respect to order of proximity is the convergence of $l^{(N)}$ to zero, defined by the condition:

I. Whatever $k$ may be, there exists an $N(k)$ such that

$$(l^{(N)}, f) = (0, 0, \dots, 0, f_{k+1}, f_{k+2}, \dots). \tag{7}$$

For operators bounded in norm,

$$\left\|(l^{(N)}, f)\right\|_X < K\|f\|_B.$$

convergence by order of closeness is a special case of weak convergence.

In the linear space of countable vectors, instead of a norm we introduce a topology by defining neighborhoods of zero $\mathfrak{B}_a$ as vectors of the form

$$(0, 0, 0, \dots, 0, a_{a+1}, a_{a+2}, \dots) \tag{8}$$

with the corresponding topology $f$.

$$B_a = (0, 0, \dots, 0, f_{a+1}, f_{a+2}, \dots). \tag{9}$$

In this sense convergence by order of closeness will be uniform convergence. To each prescribed neighborhood $\mathfrak{B}_a$ one can assign such an $N$ that

$$(l^{(N)}, f) \in \mathfrak{B}_a \quad \text{for } N > N(a). \tag{10}$$

Let us consider two examples.

**Example 1.** Let $f(x, y)$ be an analytic function of two variables $(x, y)$, expandable in a power series in a neighborhood of zero:

$$f(x, y) = a_0 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \dots . \tag{11}$$

We shall regard the function $f$ as a vector-valued function $(a_0,\ \ a_{10}x + a_{01}y,\ \ a_{20}x^2 + a_{11}xy + a_{02}y^2, \ldots)$, whose components are the homogeneous polynomials of which the expansion (11) consists.

The convergence of $l^{(N)}$ to zero by order of closeness means that the formulas $(l^{(N)}, f) = 0$, for sufficiently large $N$, will be valid for polynomials of arbitrarily high degree. Introduce the transformation:

$$f_h(X, Y) = f(hX, hY).$$

The set of functions of the unit ball in $X$ that lie in a closer neighborhood of zero $\mathfrak{B}_a$, for sufficiently small $h$, will be in a closer neighborhood of zero $\|f_h\| < Ah^a$ in the sense of the topology $X$. One may say that a neighborhood in $B$ is stratified into a countable set of neighborhoods in $T$.

**Example 2.** Let the function $f(\vartheta, \varphi)$ be given on the unit sphere of three-dimensional space and belong there to $W_2^{(2)}$, i.e., for example:

$$\|f\|_{W^{(2)}2} = \iint_S (\Delta f)^2\, dS + \left(\iint_S f\, dS\right)^2. \tag{12}$$

Expand $f$ in a series in spherical harmonics:

$$f = \sum_{n=0}^{\infty} Y_n(\vartheta, \varphi) \tag{13}$$

and we shall define $f$ as an abstract function with values

$$f(Y_0,\ Y_1(\vartheta, \varphi),\ Y_2(\vartheta, \varphi), \ldots). \tag{14}$$

Convergence by order of closeness in this case will mean that, for sufficiently large $N$, $(l^{(N)}, f)$ will vanish for an arbitrarily large number of spherical harmonics.

The neighborhoods of zero $\mathfrak{B}_a$ for the values on the sphere of a certain analytic function of three variables $f(x, y, z)$, under the substitution

$$f_h(X, Y, Z) = f(hX, hY, hZ),$$

are again stratified into $\varepsilon$-neighborhoods in the topology $B$, with the peculiarity that some $f$ may accidentally fall into a closer neighborhood of zero than $Ah^a$.

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## CITED LITERATURE

[1] S. M. Nikol' skii, *Quadrature Formulas*, Moscow, 1958. [2] S. L. Sobolev, DAN, **137**, No. 3, 527 (1961).

*Note: Figure translations are in progress. See original paper for figures.*