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Abstract

Full Text

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ON THE EXTENSION OF AN OPERATOR AND THE EXISTENCE OF FIXED POINTS

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Let E be a Banach space. We shall denote by $\theta \in E$ the zero element. We consider operators acting from E into E.

The following is known (^{1}).

Theorem 1. Let F be a closed set; let A be a completely continuous operator defined on F; let T be the closure of the convex hull $A(F)$, and let $\varepsilon > 0$. Then there exists a completely continuous operator A^ε, defined on the whole space E, for which

$$A^\varepsilon x = Ax \qquad \text{for } x \in F; \tag{1}$$

$$\rho(A^\varepsilon x, T) < \varepsilon \qquad \text{for } x \in E. \tag{2}$$

In (^{1}) the question is raised of the possibility of constructing an operator $\widetilde{A}$ such that

$$\widetilde{A}x = Ax \qquad \text{for } x \in F; \tag{3}$$

$$\widetilde{A}(E) \subset T. \tag{4}$$

A positive answer is given by

Theorem 2. Under the conditions of Theorem 1 there exists a completely continuous operator $\widetilde{A}$, defined on the whole space E and satisfying conditions (3), (4).

Proof. Take a number $\varepsilon > 0$. Let the operator A^ε satisfy the requirements of Theorem 1. Consider the closed ball $\overline{O}_R$ of radius $R > 0$ with center at the point θ. Let $\eta_1, \eta_2, \dots, \eta_p$ be an ε_1-net for $A^\varepsilon(\overline{O}_R)$, with some $\varepsilon_1 > 0$. For each point η_i choose a point $y_i \in T$ so that

$$\rho(\eta_i, y_i) < \rho(\eta_i, T) + \varepsilon_1.$$

Define on $\overline{O}_R$ the operator by the equality

$$A_1 x = \sum_{i=1}^{p} \mu_i(x) y_i \Big/ \sum_{i=1}^{p} \mu_i(x),$$

where

$$\mu_i(x) = \begin{cases} 2\varepsilon_1 - \|A^\varepsilon x - \eta_i\|, & \text{for } \|A^\varepsilon x - \eta_i\| \leq 2\varepsilon_1, \\ 0, & \text{for } \|A^\varepsilon x - \eta_i\| > 2\varepsilon_1. \end{cases}$$

Next, let on the number line a function be defined for $z \geq 0$ by

$$f(z) = \begin{cases} z/\varepsilon_1, & \text{for } 0 \leq z \leq \varepsilon_1, \\ 1, & \text{for } z > \varepsilon_1. \end{cases}$$

Define the operator B_1 on the set $P_1 = F + \overline{O}_R$ in the following way:

$$B_1 x = \begin{cases} A^\varepsilon x + f[\rho(A^\varepsilon x, T)](A_1 x - A^\varepsilon x), & \text{for } x \in \overline{O}_R - F, \\ Ax, & \text{for } x \in F. \end{cases}$$

The operator B_1 is completely continuous on P_1, and $B_1 x = Ax$ for $x \in F$. It turns out that for $\varepsilon_1 < \varepsilon/14$ one has

$$\rho(B_1 x, T) < \frac{\varepsilon}{2} \quad \text{for } x \in P_1,$$

$$\|B_1 x - A^\varepsilon x\| < 2\varepsilon \quad \text{for } x \in P_1.$$

Applying our construction to the operator B_1, we construct the operator B_2, and so on. We obtain a sequence of completely continuous operators $\{B_n\}$, defined on P_1 and satisfying, for $n = 1, 2, \dots, \infty$, the conditions

$$B_{nx} = Ax \quad \text{for } x \in F,$$

$$\rho(B_{nx}, T) < \varepsilon/2^n \quad \text{for } x \in P_1,$$

$$\|B_{nx} - B_{n-1}x\| < \varepsilon/2^{n-2} \quad \text{for } x \in P_1.$$

It is easy to show that $\{B_n\}$ converges on P_1 to a completely continuous operator C_1, for which

$$C_1 x = Ax \quad \text{for } x \in F,$$

$$\rho(C_1 x, T) = 0 \quad \text{for } x \in P_1.$$

Starting from the operator C_1, construct an operator C_2, defined on the set $P_2 = F + \overline{O}_{2R}$, where $\overline{O}_{2R}$ is the ball of radius $2R$ with center at the point θ, and so on. We obtain a sequence of completely continuous operators $\{C_n\}$ on the sets $P_n = F + \overline{O}_{2^{nR}}$ such that

$$C_{nx} = C_{n-1}x \quad \text{for } x \in P_{n-1},$$

$$\rho(C_{nx}, T) = 0 \quad \text{for } x \in P_n.$$

The operator $\widetilde{A}$ is defined on E by the equality

$$\widetilde{A}x = C_{n_0}x,$$

where n_0 is the least n among those satisfying the inequality $2^{nR} \geq \|x\|$. $\widetilde{A}$ satisfies all the requirements of Theorem 2.

Remark. Theorems 1 and 2 remain valid if the domain of definition of the operator A belongs to an arbitrary metric (not necessarily Banach) space. The essential point is that the values of the operator A belong to a Banach space.

Let Γ be the boundary of an open bounded set G, and let the operator A be completely continuous on $\overline{G}$ and have no fixed points on the boundary Γ. Then the topological degree (3) $d(\Phi, G, \theta)$ is defined for the mapping $\Phi x = x - Ax$ at the point θ on the set G. It is known (3) that if $d(\Phi, G, \theta) \neq 0$, then the operator A has a fixed point.

Another criterion for the existence of a fixed point is due to Schauder (4):

If a continuous operator A maps a convex closed set F into a compact set $\Delta \subset F$, then the operator A has a fixed point in F.

Theorem 2 makes it possible to prove this assertion by means of the notion of topological degree. Let T be the closure of the convex hull of the set $A(\Delta)$. Let the operator $\widetilde{A}$ satisfy all the conditions of Theorem 2

and G is a bounded open set containing Δ. Then for $\widetilde{\Phi}x = x - \widetilde{A}x$ we have $d(\widetilde{\Phi}, G, \theta) = 1$, which proves the required assertion. Denote by K the cone in the space E [5].

Theorem 3. *Let Γ be the boundary of an open bounded set G, $\theta \in G$, and let the operator A be completely continuous on the set $F = K\Gamma$, with $A(F) \subset K$, $\rho(\theta, A(F)) > 0$. Then there exists an element $x_0 \in F$ such that $Ax_0 = \lambda x_0$ and $\lambda > 0$.*

Theorem 3 was proved in [1]. Theorem 2 makes it possible to obtain a simple proof of it.

Let $u_0 \in K$ and $u_0 \neq \theta$. Consider the completely continuous operator

$$Bx = \widetilde{A}x + \rho[\widetilde{A}x, A(F)]u_0,$$

where the operator $\widetilde{A}$ satisfies the requirements of Theorem 2. It turns out that

$$\inf_{x \in G} Bx > 0, \qquad Bx = Ax \quad \text{for } x \in F.$$

Then there exists an element $x_0 \in \Gamma$ [1] such that $Bx_0 = \lambda x_0$ and $\lambda > 0$, whence Theorem 3 follows.

Theorem 4. *Let* Γ *be the boundary of an open bounded set* G, $\theta \in G$. *Suppose, further, that the operator* A *is completely continuous on the set* $F = K \cap \overline{G}$, *and that for* $x \in K \cap \Gamma$ *the equality* $\varphi = \lambda A\varphi$ $(\lambda > 0)$ *implies* $\lambda \geq 1$. *Then the operator* A *has a fixed point on* F.

Theorem 5. *Let* Γ_1, Γ_2 *be, respectively, the boundaries of open bounded sets* G_1, G_2; $\theta \in G_1$, $\theta \in G_2$, $G_1 \neq G_2$. *Suppose, further, that the operator* A *is completely continuous on the set* $F = (\overline{G}_1 + \overline{G}_2 - G_1G_2)K$, $A(F) \in K$, *and that for* $x \in K\Gamma_1$, *from the condition* $\varphi = \lambda A\varphi$ $(\lambda > 0)$ *it follows that* $\lambda \geq 1$, *while for the set* $F_2 = K\Gamma_2$ *there is an element* $u_0 \in K$ $(u_0 \neq \theta)$ *such that* $x - Ax \neq tu_0$ *for* $t > 0$, $x \in F_2$. *Then the operator* A *has a fixed point on* F.

Theorems 4 and 5 are proved by means of Theorem 2 by a topological method and are generalizations of Theorems 1 and 2 of [2].

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CITED LITERATURE

1. M. A. Krasnosel' skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Moscow, 1956.

2. M. A. Krasnosel' skii, DAN, 135, No. 3 (1960).

3. J. Leray, J. Schauder, UMN, 1, 3-4 (13-14) (1946).

4. V. V. Nemytzkii, Mat. sbornik, 41, 4 (1934).

5. M. G. Krein, M. A. Rutman, UMN, 3, 23 (1948).

Note: Figure translations are in progress. See original paper for figures.

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