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# MATHEMATICS

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**Abstract**

**Full Text**

## MATHEMATICS

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# ON THE UNICELLULARITY OF REAL VOLTERRA OPERATORS

*(Presented by Academician V. I. Smirnov, 1 VI 1962)*

Let an involution  $S$  be given in a separable Hilbert space  $\mathfrak{H}$ . A subspace  $\mathfrak{H}_0 \subset \mathfrak{H}$  invariant with respect to  $S$  is called  **$S$ -real**. If  $A$  is a certain bounded linear operator in  $\mathfrak{H}$  and  $AS = SA$ , then the operator  $A$  is called  **$S$ -real**.

Let the  $S$ -real operator  $A$  be a **Volterra** operator, i.e. completely continuous and having no spectral points distinct from zero. It is called **unicellular** <sup>(1)</sup> if the set of all its invariant subspaces is ordered by inclusion, and  **$S$ -unicellular** if the set of its  $S$ -real invariant subspaces is ordered by inclusion. In the finite-dimensional case the notions of unicellularity and  $S$ -unicellularity are equivalent.

In the present article an analytic criterion for  $S$ -unicellularity is established for  $S$ -real Volterra operators. From it, in particular, it follows that in an infinite-dimensional space there exist non-unicellular  $S$ -real Volterra operators which, however, are  $S$ -unicellular.

1. Let a bounded linear operator  $A$  be given in the space  $\mathfrak{H}$ , with completely continuous imaginary component

$$\frac{A - A^*}{2i}.$$

There exists a completely continuous linear mapping  $R$  of some Hilbert space  $\mathfrak{H}_W$  into the space  $\mathfrak{H}$ , and an operator  $J$  ( $J = J^*$ ,  $J^2 = E$ ) acting in  $\mathfrak{H}_W$ , such that

$$\frac{A - A^*}{2i} = RJR^*$$

and the set of vectors of the form  $A^n Rf$  ( $n = 0, 1, \dots$ ;  $f \in \mathfrak{H}_W$ ) is complete in  $\mathfrak{H}$ . The operator-function

$$W(\lambda) = E - 2iR^*(A - \lambda E)^{-1}RJ \tag{1}$$

is called the **characteristic** <sup>(2-4)</sup> function of the operator  $A$ . The characteristic function (1) will be called  **$T$ -real** if in the space  $\mathfrak{H}_W$  there exists an involution  $T$  for which  $TW(\bar{\lambda})T = W(\lambda)$  and  $TJT = -J$ . It is easy to show that every  $S$ -real operator  $A$  has such a  $T$ -real characteristic function  $W(\lambda)$ , and that the involutions  $S$  and  $T$  are connected by the relation  $SR = RT$ . The converse assertion is also true:

**Theorem 1.** *If the operator  $A$  has a  $T$ -real characteristic function (1), then in the space  $\mathfrak{H}$  there exists an involution  $S$  such that  $AS = SA$  and  $SR = RT$ .*

**Proof.** From the definition of the characteristic function it follows that

$$W^*(\mu)JW(\lambda) - J = 4 \frac{\lambda - \bar{\mu}}{2} JR^*(A^* - \bar{\mu}E)^{-1}(A - \lambda E)^{-1}RJ.$$

Consequently,

$$TR^*(A^* - \bar{\mu}E)^{-1}(A - \lambda E)^{-1}RT = R^*(A^* - \bar{\mu}E)^{-1}(A - \bar{\lambda}E)^{-1}R,$$

$$(A^{nRTf}, A^{mRTg}) = \overline{(A^{nRf}, A^{mRg})} \quad (n, m = 0, 1, \dots; f, g \in \mathfrak{H}_W).$$

Define the required involution  $S$  on the dense set of vectors in  $\mathfrak{H}$  by setting

$$S \left( \sum_{k=0}^n A^k R g_k \right) = \sum_{k=0}^n A^k RT g_k \quad (n = 0, 1, \dots; g_k \in \mathfrak{H}_W).$$

2. Consider an  $S$ -real operator  $A$  and its  $T$ -real characteristic function (1), satisfying the condition  $SR = RT$ . To each subspace  $\mathfrak{H}_0 \subset \mathfrak{H}$  invariant with respect to  $A$  there corresponds the operator-function

$$W_0(\lambda) = E - 2iR_0^*(A_0 - \lambda E)^{-1}R_0J \quad (R_0 = P_0R), \quad (2)$$

where  $P_0$  is the projection operator onto  $\mathfrak{H}_0$  and  $A_0f = P_0Af$  ( $f \in \mathfrak{H}_0$ ). The function  $W_0(\lambda)$  is characteristic for the operator  $A_0$  and is called the **projection** of the function  $W(\lambda)$  onto the subspace  $\mathfrak{H}_0$ .

**Theorem 2.** For an invariant subspace  $\mathfrak{H}_0$  with respect to  $A$  to be  $S$ -real, it is necessary and sufficient that the projection (2) satisfy the condition

$$TW_0(\bar{\lambda})T = W_0(\lambda).$$

3. The characteristic function  $W(\lambda)$  of a Volterra operator has the following properties:

- 1)  $W(\lambda) - E$  is an entire function of  $\frac{1}{\lambda}$ , taking completely continuous values and satisfying the condition  $W(\infty) = E$ ;
- 2)

$$W^*(\lambda)JW(\lambda) - J \geq 0 \quad (\text{Im } \lambda > 0), \quad W^*(\lambda)JW(\lambda) - J = 0 \quad (\text{Im } \lambda = 0).$$

Denote by  $\Omega_J$  the set of all operator-functions possessing properties 1) and 2).  
If

$$W_1(\lambda) = W_2(\lambda)W_3(\lambda) \quad (W_k(\lambda) \in \Omega_J),$$

then we shall say that  $W_2(\lambda)$  is a **left divisor** of the function  $W_1(\lambda)$ , and that the function  $W_2(\lambda)$  **precedes** the function  $W_1(\lambda)$ .

Consider an  $S$ -real Volterra operator  $A$  and its  $T$ -real characteristic function  $W(\lambda)$ . If  $SR = RT$ , then every projection of the function  $W(\lambda)$  onto an  $S$ -real invariant subspace of the operator  $A$  is a  $T$ -real left divisor of the function  $W(\lambda)$ . The converse assertion is false in general.

**Theorem 3.** Let  $A$  be a Volterra  $S$ -real operator, let  $W(\lambda)$  be its  $T$ -real characteristic operator-function, and let  $SR = RT$ . For the operator  $A$  to be  $S$ -unicellular, it is necessary and sufficient that the set of all left  $T$ -real divisors of the function  $W(\lambda)$  be ordered.

4. Consider an  $S$ -real simple (1) Volterra operator  $A$  with a finite-dimensional imaginary component. Since the spectrum of this component is symmetric with respect to the origin, in the space

$$\frac{A - A^*}{2i} \mathfrak{H}$$

there exists an orthonormal basis

$$\{e_1^+, \dots, e_n^+, e_1^-, \dots, e_n^-\}$$

such that

$$Se_\alpha^+ = e_\alpha^-, \quad \frac{A - A^*}{2i} e_\alpha^+ = \omega_\alpha e_\alpha^+ \quad (\omega_\alpha > 0), \quad \frac{A - A^*}{2i} e_\alpha^- = -\omega_\alpha e_\alpha^-.$$

In an arbitrary  $2n$ -dimensional unitary space  $\mathfrak{H}_W$ , choose an orthonormal basis

$$\{h_1^{(1)}, \dots, h_n^{(1)}, h_1^{(2)}, \dots, h_n^{(2)}\}$$

and define linear operators  $J$  and  $R$  by setting

$$Jh_\alpha^{(1)} = h_\alpha^{(1)}, \quad Jh_\alpha^{(2)} = -h_\alpha^{(2)}, \quad Rh_\alpha^{(1)} = \omega_\alpha^{1/2} e_\alpha^+, \quad Rh_\alpha^{(2)} = \omega_\alpha^{1/2} e_\alpha^-.$$

In addition, define an involution  $T$  in the space  $\mathfrak{H}_W$ , specifying it by the equalities  $Th_\alpha^{(1)} = h_\alpha^{(2)}$ . Then  $SR = RT$ , and

$$W(\lambda) = E - 2iR^*(A - \lambda E)^{-1}RJ$$

is a  $T$ -real characteristic operator-function of the operator  $A$ . Consider the matrix

$$V(\lambda) = \begin{pmatrix} v_{11}(\lambda) & v_{12}(\lambda) \\ v_{21}(\lambda) & v_{22}(\lambda) \end{pmatrix} \quad \left( v_{ij}(\lambda) = \|(W(\lambda)h_\alpha^{(i)}, h_\beta^{(j)})\|_{\alpha, \beta=1}^n \right). \quad (3)$$

The condition  $TW(\bar{\lambda})T = W(\lambda)$  is equivalent to the condition

$$v_{11}(\lambda) = \overline{v_{22}(\bar{\lambda})}, \quad v_{12}(\lambda) = \overline{v_{21}(\bar{\lambda})}. \quad (4)$$

By virtue of Theorem 3, the operator  $A$  is then and only then  $S$ -unicellular when the set of all right divisors of the matrix function  $V(\lambda)$  satisfying conditions of the form (4) is ordered.

5. In the space  $\mathfrak{L}_2^{(2)}[0, 1]$ , whose elements are vector functions  $f(x) = \|f_1(x) \ f_2(x)\|$  ( $0 \leq x \leq 1$ ) and in which the scalar product is defined by the equality

$$(f(x), g(x)) = \int_0^1 f(t)g^*(t) dt,$$

let us define a simple Volterra operator\*

$$A(\|f_1(x) \ f_2(x)\|) = \int_x^1 \|f_1(t) \ f_2(t)\| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} dt. \quad (5)$$

It is obvious that the operator  $A$  commutes with the involution

$$S(\|f_1(x) \ f_2(x)\|) = \|\overline{f_1(x)} \ \overline{f_2(x)}\|.$$

It is not unicellular, since the sets of vectors of the form

$$\|f_1(x) \ if_1(x)\| \quad \text{and} \quad \|f_1(x) \ -if_1(x)\|$$

are invariant, mutually orthogonal subspaces with respect to  $A$ . A direct calculation shows that the imaginary component of the operator  $A$  is two-dimensional and that the matrix (3) has the form

$$V(\lambda) = \left\| \begin{array}{cc} e^{i/\lambda} & 0 \\ 0 & e^{-i/\lambda} \end{array} \right\|.$$

Let

$$V(\lambda) = V_1(\lambda)V_2(\lambda),$$

where

$$V_2(\lambda) = \left\| \begin{array}{cc} \frac{a(\lambda)}{b(\bar{\lambda})} & \frac{b(\lambda)}{a(\bar{\lambda})} \\ b(\bar{\lambda}) & a(\bar{\lambda}) \end{array} \right\|$$

is a right divisor of the matrix  $V(\lambda)$  satisfying conditions (4). From the equalities

$$V_k(\lambda)JV_k^*(\lambda) - J \geq 0 \quad \left( J = \left\| \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right\|, \operatorname{Im} \lambda \geq 0, k = 1, 2 \right)$$

it follows easily that

$$a(\lambda)\overline{a(\bar{\lambda})} - b(\lambda)\overline{b(\bar{\lambda})} \equiv 1, \quad a(\lambda) = 1 + \frac{i\mu}{\lambda} + \dots \quad (\mu > 0) \quad (6)$$

and that the function  $a(\lambda)$  has no zeros. Consequently,

$$a(\lambda) = e^{g(\lambda)},$$

where  $g(\lambda)$  is an entire function of  $1/\lambda$ . Moreover, by virtue of the multiplicative representation

$$V_2(\lambda) = \int_0^1 e^{\frac{t}{\lambda} dE(t)J} \quad (2)$$

there is the estimate

$$|e^{g(\lambda)}| \leq e^{\sigma/\lambda} \quad (\sigma > 0),$$

and therefore

$$g(\lambda) = \alpha + \beta/\lambda.$$

Using formulas (6), we obtain:

$$\alpha = 0, \quad \beta = i\mu, \quad a(\lambda) = e^{i\mu/\lambda}, \quad b(\lambda) = 0.$$

Thus,

$$V_2(\lambda) = \left\| \begin{array}{cc} e^{i\mu/\lambda} & 0 \\ 0 & e^{-i\mu/\lambda} \end{array} \right\| \quad (0 < \mu < 1).$$

This proves the orderedness of the set of all right divisors of the matrix function  $V(\lambda)$  satisfying conditions (4), i.e., the  $S$ -unicellularity of the operator (5) is proved.

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\* Several years ago M. G. Krein expressed the hypothesis of the  $S$ -unicellularity of all  $S$ -real simple Volterra operators with two-dimensional imaginary component. Having learned of the criterion of  $S$ -unicellularity formulated in this article, M. G. Krein recommended that the author apply it to example (5).

*Note: Figure translations are in progress. See original paper for figures.*

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