ANALYTIC PROPERTIES OF PARTIAL WAVES OF THE SCATTERING AMPLITUDE IN PERTURBATION THEORY

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Abstract

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D. Ya. PETRINA

ANALYTIC PROPERTIES OF PARTIAL WAVES OF THE SCATTERING AMPLITUDE IN PERTURBATION THEORY

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For the study of the analytic properties of the partial waves of the scattering amplitude, the Mandelstam representation and the unitarity condition are used ([1−4]). From the unitarity condition and the Mandelstam spectral representation one can draw conclusions concerning the analytic properties of partial waves both on the first and on the second sheet of the Riemann surface; in particular, one can infer the possibility of the existence of poles on the second sheet.

However, the Mandelstam representation has so far not been rigorously proved in perturbation theory, and its validity can only be assumed, while the unitarity condition is in fact used only on the elastic energy interval. It should also be noted that in order to draw conclusions concerning the analytic properties of partial waves it is sufficient to know the analytic properties of the scattering amplitude with respect to the energy for $\cos\theta$ (the cosine of the scattering angle) in the interval $-1 \leq \cos\theta \leq 1$, and not a complete description of its analytic properties in all variables, which is contained in the Mandelstam representation.

It is therefore natural to pose the problem of investigating partial waves on the basis of the general properties of the scattering amplitude in perturbation theory, without using the assumption of the validity of the Mandelstam representation; moreover, one must answer the question of whether poles actually appear on the second sheet.

1. The scattering amplitude in perturbation theory consists of contributions from Feynman diagrams, which can be represented in the following form (the scattering of scalar particles of equal masses and strongly connected diagrams are considered; we assume that three lines enter each vertex) ([5,6]):

$$F(s,t) = \int \frac{c(\alpha)\,\delta\big(\sum_{i=1}^{n}\alpha_i - 1\big)\; d\alpha_1 \dots d\alpha_n}{f(\alpha)s + g(\alpha)t - m^2 k(\alpha)}, \qquad \alpha_1 \geq 0, \dots, \alpha_n \geq 0. \quad (1)$$

Here s and t are the Mandelstam variables; m is the mass of the internal and external particles. (In fact, the contribution in general form will differ from expression (1) in that the denominator $f(\alpha)s+g(\alpha)t-m^2k(\alpha)=D(\alpha,s,t,m^2)$ may also occur in a power higher than the first; however, this power does not affect the analytic properties of the contribution, and therefore we shall investigate expression (1).)

It is known that the functions $f(\alpha)$, $g(\alpha)$, $k(\alpha)$, $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$, are real homogeneous polynomials and possess the property that, for s and t in the domain

$$s<4m^2,\qquad t<4m^2,\qquad u=4m^2-s-t<4m^2 \tag{2}$$

the denominator $D(\alpha,s,t,m^2)$ nowhere vanishes in the region of integration ([7,8]). This fact is decisive in the study of the analytic properties of partial waves.

Theorem 1. *A partial wave of the scattering amplitude in perturbation theory is an analytic function of s in the complex plane*

with cuts

$$\operatorname{Im} s=0,\qquad \operatorname{Re} s\leqslant 0,\qquad \operatorname{Re} s\geqslant 4m^2. \tag{3}$$

Proof. From the expansion of $F(s,t)$ in a series in partial waves

$$F(s,t)=F(s,\cos\theta)=\sum_{l=0}^{\infty}(2l+1)f_l(s)P_l(\cos\theta),\qquad \cos\theta=1+\frac{2t}{s-4m^2}$$

(P_l is a Legendre polynomial), it follows that $f_l(s)$ is represented in the form

$$f_l(s)=\frac{1}{2}\int c(\alpha)\,\delta\left(\sum_{i=1}^{n}\alpha_i-1\right)\times\left\{\int_{-1}^{1}\frac{P_l(z)\,dz}{f(\alpha)s+g(\alpha)\frac{z-1}{2}(s-4m^2)-m^2k(\alpha)}\right\}d\alpha_1\dots d\alpha_n. \tag{4}$$

We shall next show that for $0<s<4m^2$, $-1\leqslant z\leqslant 1$ the denominator

$$D\left(\alpha,s,\frac{z-1}{2}(s-4m^2),m^2\right)$$

does not vanish in the domain of integration. It is easy to verify that for $0<s<4m^2$, $-1\leqslant z\leqslant 1$ the inequalities

$$0\leqslant\frac{z-1}{2}(s-4m^2)<4m^2,\qquad 4m^2-s-\frac{z-1}{2}(s-4m^2)<4m^2$$

hold, i.e., the quantities $s,\ t = \frac{z-1}{2}(s-4m^2)$ lie in the domain (2), and, consequently, the denominator

$$D\left(\alpha, s, \frac{z-1}{2}(s-4m^2), m^2\right)$$

does not vanish for all α from the domain of integration.

The denominator

$$D\left(\alpha, s, \frac{z-1}{2}(s-4m^2), m^2\right)$$

will not vanish also for s from the complex plane with cuts (3). Indeed,

$$D\left(\alpha, s, \frac{z-1}{2}(s-4m^2), m^2\right)$$

for complex s can vanish only on that set of z and α which is determined by the equality

$$f(\alpha) + \frac{z-1}{2} g(\alpha) = 0.$$

For these values of α and z the denominator

$$D\left(\alpha, s, \frac{z-1}{2}(s-4m^2), m^2\right)$$

does not depend on s and is equal to

$$-\frac{z-1}{2}\, 4m^2 g(\alpha) - m^2 k(\alpha).$$

But the denominator is equal to the same expression on this set also for $0 < s < 4m^2$. Since for such s the denominator does not vanish anywhere for $-1 \leqslant z \leqslant 1$ and α from the domain of integration, the expression

$$-\frac{z-1}{2}\, 4m^2 g(\alpha) - m^2 k(\alpha)$$

does not vanish on the set

$$f(\alpha) + \frac{z-1}{2} g(\alpha) = 0.$$

Thus, for all s from the complex plane with cuts (3), the denominator

$$D\left(\alpha, s, \frac{z-1}{2}(s-4m^2), m^2\right)$$

does not vanish. Consequently, the partial wave $f_l(s)$ is an analytic function in the complex plane with cuts (3).

Remark. In fact, we have established that for $-1 \leqslant \cos\theta \leqslant 1$ the scattering amplitude is an analytic function of s in the complex plane with cuts (3), while for $0 < s < 4m^2$ the scattering amplitude is an analytic function of $\cos\theta$ in the complex plane with cuts

$$\operatorname{Im}\cos\theta = 0, \qquad |\operatorname{Re}\cos\theta| > 1.$$

This fact makes it possible to derive dispersion relations in the energy at fixed cosine of the scattering angle and in the cosine of the scattering angle at fixed energy*.

* After the present work had been prepared for publication and reported at the seminar

on quantum field theory at the Institute of Mathematics of the Academy of Sciences of the Ukrainian SSR,

we learned of an analogous result of Taylor ([9]).

2. We shall now consider in detail the unitarity condition in perturbation theory. From the unitarity condition it is easy to obtain the following relation ($f_{nl}(s)$ is the partial wave of the scattering amplitude of n-th order)

$$\operatorname{Im} f_{nl}(s) = \frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}} \sum_i f_{n-i\,l}(s) f^*_{il}(s), \qquad 4m^2 \leqslant s < 9m^2, \tag{5}$$

where the summation over i in formula (5) is carried out over all possible divisions of the Feynman diagram of n-th order into two parts, of $(n-i)$-th and i-th order, in such a way that these parts are connected by two lines and these two lines carry momentum $\sqrt{s}$. On the basis of equality (5) one may make the following assertion.

Theorem 2. If all $f_{jl}(s)$ entering formula (5) have the form

$$f_{jl}(s) = F_{jl}(s) + i\frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\, G_{jl}(s) \qquad (j < n), \tag{6}$$

where the functions $F_{jl}(s)$ and $G_{jl}(s)$ are analytic in the complex plane with cuts

$$\operatorname{Im} s = 0, \qquad \operatorname{Re} s \leqslant 3m^2, \qquad \operatorname{Re} s \geqslant 9m^2 \tag{7}$$

and on the interval $4m^2 \leqslant s < 9m^2$ the functions $F_{jl}(s)$ and $G_{jl}(s)$ are real, then the function $f_{nl}(s)$ will have the form (6), and the functions $F_{nl}(s)$ and $G_{nl}(s)$ will possess the same properties as $F_{jl}(s)$ and $G_{jl}(s)$.

(We note that $F_{jl}(s)$ and $G_{jl}(s)$ in expression (6) must form a function analytic in the same domain as the function $f_{jl}(s)$, i.e., in the complex plane with cuts (3).)

Proof. Let us introduce for consideration the function

$$F_{nl}(s) = f_{nl}(s) - i\frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\sum_i f_{n-i\,l}(s) f^*_{il}(s) = f_{nl}(s) - i\frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\, G_{nl}(s). \tag{8}$$

For $4m^2 \leqslant s < 9m^2$ the functions $F_{jl}(s)$ and $G_{jl}(s)$ are real; therefore

$$f^*_{jl}(s) = F_{jl}(s) - i\frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\, G_{jl}(s),$$

and, consequently, $f^*_{jl}(s)$ is an analytic function in the complex plane with cuts

$$\operatorname{Im} s = 0, \qquad \operatorname{Re} s \leqslant 3m^2, \qquad \operatorname{Re} s \geqslant 4m^2. \tag{9}$$

It should be noted that $f^*_{jl}(s)$ differs in the sign of the second term from $f_{jl}(s)$, and therefore its domain of analyticity will consist of the intersection of the domains of analyticity of $F_{jl}(s)$ and $\frac{\sqrt{s-4m^2}}{\sqrt{s}} G_{jl}(s)$, or, what is the same thing, of the intersection of the domains of analyticity of $f_{jl}(s)$ and $\frac{\sqrt{s-4m^2}}{\sqrt{s}} G_{jl}(s)$. The function

$$G_{nl}(s) = \sum_i f_{n-i\,l}(s) f^*_{il}(s)$$

is analytic in the complex plane with cuts (9). Since

$$\lim_{\varepsilon \to +0} f_{jl}(s + i\varepsilon) = \lim_{\varepsilon \to +0} f^*_{jl}(s - i\varepsilon)$$

on the cuts (9), then also

$$\lim_{\varepsilon \to +0} G_{nl}(s + i\varepsilon) = \lim_{\varepsilon \to +0} G^*_{nl}(s - i\varepsilon)$$

on the cuts (9). For $4m^2 \leqslant s < 9m^2$ the function $G_{nl}(s)$ is real by virtue of formula (5) and, consequently, analytic on this interval. Hence one may conclude that the function $G_{nl}(s)$ is analytic in the complex plane with cuts (7). Since $F_{nl}(s)$, by virtue of formula (8), is analytic in the complex plane with cuts (9), for the complete proof of the theorem it remains to show that the function $F_{nl}(s)$ is analytic also on the interval $4m^2 \leqslant s < 9m^2$. Since

$$\lim_{\varepsilon\to+0} F_{nl}(s+i\varepsilon) = \lim_{\varepsilon\to+0} F^*_{nl}(s-i\varepsilon)$$

on this interval, then it suffices for us to show that $F_{nl}(s)$ is a real function on this interval. From formula (8) we have

$$\operatorname{Im} F_{nl}(s) = \operatorname{Im} f_{nl}(s) - \frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\, G_{nl}(s) = 0, \qquad 4m^2 \leqslant s < 9m^2,$$

i.e. $F_{nl}(s)$ is a real function on this interval. The theorem is proved.

Remark. If $f_{nl}(s)$ has the form (6), then on the second sheet of the Riemann surface, to which we pass from the first through the segment $\operatorname{Im} s = 0,\ 4m^2 \leqslant \operatorname{Re} s < 9m^2$, it will have the form

$$f^{(2)}_{nl}(s) = F_{nl}(s) - i\frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\, G_{nl}(s),$$

i.e. it will have no complex singularities on the second sheet. One can verify that the conditions of Theorem 2 are satisfied by the partial waves of the scattering amplitude in the ladder approximation and by the partial waves for any term of the expansion of the scattering amplitude on the potential [10] in a series in the parameter.

The analytic properties of the partial waves on the second sheet are determined to a significant extent by the character of the weight function $\rho(s,t)$ in the Mandelstam representation. This assertion is confirmed by the following theorem, which we state without proof.

Theorem 3. *If the weight function $\rho(s,t)$ for $t \geqslant 4m^2$ is analytic in the complex plane with the cut $\operatorname{Im} s = 0,\ \operatorname{Re} s \leqslant 0,\ \operatorname{Re} s \geqslant 4m^2$, and the function*

$$\frac{1}{\sqrt{s-4m^2}}\,\rho(s,t)$$

is analytic for $\operatorname{Im} s = 0,\ 4m^2 \leqslant \operatorname{Re} s < 9m^2$, then the partial waves have no complex singularities on the second sheet of the Riemann surface.

Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

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Note: Figure translations are in progress. See original paper for figures.