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## Abstract

## Full Text

Mechanics

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# ON THE INFLUENCE OF THE ELASTICITY OF THE SUPPORTS OF THE AXIS OF THE OUTER RING OF A CARDAN SUSPENSION ON NUTATIONAL OSCILLATIONS AND GYROSCOPE DRIFT

*(Presented by Academician A. Yu. Ishlinskii, 24 VI 1961)*

**1°.** The equations of motion by inertia of a balanced gyroscope in a cardan suspension mounted on a fixed base have the form

$$\left[A_2 + (A + A_1)\cos^2\beta + C_1\sin^2\beta\right]\ddot{\alpha} + H\dot{\beta}\cos\beta -$$
$$-2(A + A_1 - C_1)\dot{\alpha}\dot{\beta}\sin\beta\cos\beta = 0;$$
$$(A + B_1)\ddot{\beta} + (A + A_1 - C_1)\dot{\alpha}^2\sin\beta\cos\beta - H\dot{\alpha}\cos\beta = 0,$$
$$\frac{dH}{dt} = C\frac{d}{dt}(\dot{\gamma} + \dot{\alpha}\sin\beta) = 0;$$

here $\alpha, \beta, \gamma$ are the angles of rotation of the outer ring relative to the fixed base, of the inner ring relative to the outer one, and of the gyroscope relative to the inner ring, respectively; $A_2$ is the moment of inertia of the outer ring about its axis of rotation; $A_1, B_1$, and $C_1$ are the moments of inertia of the inner ring; $A$ and $C$ are the equatorial and polar moments of inertia of the gyroscope, and $H$ is its kinetic moment. These equations were investigated in detail by E. L. Nikolai (1).

The equations of motion admit a particular solution corresponding to stationary rotation of the gyroscope about an axis fixed relative to the base:

$$\alpha = \alpha_0, \qquad \beta = \beta_0, \qquad \dot{\alpha} = 0, \quad \dot{\beta} = 0, \qquad H = C\omega = \text{const.}$$

If the outer ring is impulsively given an angular velocity $\Omega \ll \omega$, then the gyroscope will execute nutational oscillations

$$\alpha = \alpha_0 + \frac{\Omega}{\nu_0}\sin\nu_0 t + O(\Omega^2), \qquad \beta = \beta_0 + \frac{I_0\Omega}{H\cos\beta_0}(1 - \cos\nu_0 t) + O(\Omega^2)$$

with frequency $\nu_0 = H\cos\beta_0/\sqrt{(A+B_1)I_0}$, accompanied by rotation of the figure axis of the gyroscope about the axis of the outer ring with angular velocity *

$$u = -\frac{(A_2+C_1)\sin\beta_0}{2H\cos^2\beta_0}\Omega^2 + O(\Omega^3);$$

here

$$I_0 = A_2 + (A+A_1)\cos^2\beta_0 + C_1\sin^2\beta_0.$$

The fact that a rapidly rotating astatic gyroscope performs motion of the type of pseudoregular precession was first established by Nikolai ($^1$).

An interesting physical explanation of this phenomenon was given by S. S. Tikhmenev ($^2$). The projection of the moment of the reaction forces in the supports of the axis of the outer ring onto the mean direction of oscillation of the axis of the inner ring has a constant component, balanced by the gyroscopic moment that arises during precession of the gyroscope with angular velocity $u$.

**2°.** To study the reactions of the supports of the axes of the cardan suspension, we introduce two coordinate systems $Ox_2y_2z_2$ and $Ox_1y_1z_1$, associated with the outer and inner rings. The axis $Ox_2$ is directed along the axis of rotation of the outer ring, the axes $Oy_2$ and $Oy_1$ coincide with the axis of rotation of the inner ring, and the axis $Oz_1$ is the axis of rotation of the gyroscope.

The moments of the support reactions can be determined by using Euler' s equations for the outer ring, the inner ring, and the gyroscope, or by the method proposed by A. I. Lur' e ($^3$). In the problem under consideration this method requires lengthy calculations; however, for us it has advantages, since, as compensation for the indicated difficulties, it allows us, by a certain generalization, to study the influence of the elasticity of the supports on the nutational oscillations and drift of the gyroscope.

Let us determine, for example, the projection of the moment of the support reactions of the axis of the outer ring onto the axis $Oz_2$. Following the method being applied, we introduce an additional degree of freedom: let the outer ring rotate about the axis $Oz_2$ through an angle $\theta$

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* The calculation of the angular velocity of precession to within small quantities of third order was carried out by K. Magnus, R. Gudstein, D. S. Pelpor, Ya. L. Lunts, and others. However, E. L. Nikolai established the condition that the initial values of the coordinates and velocities must satisfy in order for such motion to occur ($^1$), p. 472. The projection of the trajectory of the apex onto the diametral plane of the unit sphere is shown in ($^1$), Fig. 3.

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

**Fig. 1**

**Fig. 2**

relative to a third ring (see Fig. 1). The third ring is introduced as a purely kinematic element and, naturally, must be regarded as inertia-free. Denoting by $p_2, q_2, r_2$ and $p_1, q_1, r_1$ the projections of the absolute angular velocity, respectively, of the outer ring on the axes $Ox_2y_2z_2$ and of the inner ring on the axes $Ox_1y_1z_1$,

$$p_2 = \dot\alpha\cos\theta, \quad q_2 = -\dot\alpha\sin\theta, \quad r_2 = \dot\theta, \quad p_1 = \dot\alpha\cos\theta\cos\beta - \dot\theta\sin\beta,$$

$$q_1 = \dot\beta - \dot\alpha\sin\theta, \quad r_1 = \dot\alpha\cos\theta\sin\beta + \dot\theta\cos\beta,$$

and assuming that the coordinate axes are the principal central axes of inertia of the rings, we determine the kinetic energy of the system

$$2T = A_2p_2^2 + B_2q_2^2 + C_2r_2^2 + (A + A_1)p_1^2 + (A + B_1)q_1^2 + C_1r_1^2 + C(r_1 + \dot\gamma)^2.$$

From the kinetic energy we derive Lagrange' s equations, taking into account the generalized reaction of the constraint $P_{z_2}$. If in these equations we take into account the constraint condition $\theta = 0$, then three equations reduce to the equations of motion of the gyroscope by inertia, given in § 1, while the fourth determines the reaction moment $P_{z_2}$:

$$P_{z_2} = (A_2 + C_1)\ddot\alpha\tan\beta + (C_1 + B_1 - A_1)\dot\alpha\dot\beta.$$

Similarly, it can be shown that $P_{y_2} = 0$, i.e. the moment of the reaction forces in the supports of the axis of the outer ring is always directed perpendicular to the plane of this ring (Fig. 2), and the moments of the reaction forces in the supports of the axes of the inner ring and of the gyroscope can be determined.

3°. Let us now show how to find the precession rate of the gyroscope, knowing the moment of the reaction forces $P_{z_2}$. Since the existence of motion of the type of pseudoregular precession has been rigorously proved, we have the right to imagine a reference system $OXYZ$, rotating about the axis $OX$, coinciding with the axis $Ox_2$, with a constant velocity equal to the precession rate. Relative to this system the gyroscope performs oscillations of limited amplitude in the angles $\alpha$ and $\beta$. Let $\alpha'$ and $\beta'$ be the angles, respectively, between the axis $OY$ and $Oy_2$

and between the axis $Ox_1$ and its mean position; then, regarding the amplitudes of the oscillations as small, we find the linearized equations of the gyroscope’ s nutational oscillations

$$[A_2 + (A + A_1)\cos^2\overline{\beta} + C_1\sin^2\overline{\beta}]\,\ddot{\alpha}' + H\cos\overline{\beta}\,\dot{\beta}' = 0,$$

$$(A + B_1)\ddot{\beta}' - H\cos\overline{\beta}\,\dot{\alpha}' = 0,$$

where $\overline{\beta}$ is the angle between the axis $Ox_2$ and the mean position of the axis $Ox_1$. It is known that

$$\overline{\beta} = \beta_0 + \frac{I_0\Omega}{2H\cos\beta_0} + O(\Omega^2).$$

Using this circumstance in the linearized equations, we obtain the ordinary equations of the gyroscope’ s nutational oscillations, whose solutions are given in No. 1.

It is easy to determine $P_{z_2}$ to within small quantities of the second order:

$$P_{z_2} = -(A_2 + C_1)\nu_0\Omega\ \mathrm{tg}\,\beta_0\sin\nu_0 t$$

and to project $P_{z_2}$ onto the axis $OY$. Since $P_Y = -P_{z_2}\sin\alpha'$, retaining from $\sin\alpha'$ only the first term and averaging $P_Y$ over time, we obtain the constant component

$$P_Y' = \frac{1}{2}(A_2 + C_1)\,\mathrm{tg}\,\beta_0\,\Omega^2.$$

Using Fig. 2, the precession rate of the gyroscope can be determined by the usual formula *

$$U = -P_Y'/H\cos\beta_0 = -(A_2 + C_1)\sin\beta_0\,\Omega^2/2H\cos^2\beta_0.$$

4°. The described method for determining the drift rate of the gyroscope makes it possible to compute this rate simply from the known first approximation. Consider, for example, the case in which the reactions in the supports of the axis of the outer ring develop while being accompanied by noticeable displacements of the axis itself. It is then necessary to study the nature of the reaction forces. We shall assume that these forces arise due to elastic deformations of the supports and are conservative by nature. Since $\theta$ can no longer be neglected, we shall derive the equations of motion from the kinetic energy given at the beginning of No. 2, assuming that a moment acts about the axis $Oz_2$ equal to

$$M_{z_2} = U'(\theta) = -k\theta + O(\theta^2),$$

where $U(\theta)$ is the force function. The general equations of motion are very cumbersome; therefore we shall restrict ourselves to studying the linear approximation

$$[A_2 + (A + A_1)\cos^2\beta_0 + C_1\sin^2\beta_0]\,\ddot{\alpha} + H\cos\beta_0\dot{\beta}$$

$$-(A + A_1 - C_1)\sin\beta_0\cos\beta_0\ddot{\theta} = 0,$$

$$-H\cos\beta_0\dot{\alpha} + (A + B_1)\ddot{\beta} + H\sin\beta_0\dot{\theta} = 0,$$

$$-(A + A_1 - C_1)\sin\beta_0\cos\beta_0\ddot{\alpha} - H\sin\beta_0\dot{\beta}$$

$$+ [C_2 + (A + A_1)\sin^2\beta_0 + C_1\cos^2\beta_0]\ddot{\theta} + k\theta = 0.$$

We integrate these equations under the initial conditions

$$\dot{\alpha} = \Omega, \qquad \dot{\beta} = 0, \qquad \theta = 0, \qquad \dot{\theta} = 0.$$

Suppose that drift exists, and introduce the reference frame $OXYZ$, whose axis $OX$ coincides with the axis of rotation of the third ring, while the axis $OZ$ coincides with the mean position of the axis of the outer ring $Oz_2$. Project the moments applied to the outer ring onto the axis $OY$:

$$M_Y = \dot{A}\dot{\alpha}\dot{\theta} - C_2\dot{\theta}\alpha - k\theta\alpha;$$

the angular drift rate is determined by the formula

$$u' = -M_Y'/H\cos\beta_0,$$

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* If for the gyroscope $A + A_1 - C_1 = 0$, then this formula can be obtained from the condition $p_1 = 0$. Indeed, it follows from this that $\ddot{\alpha}_2 \cos\beta_0 = -\dot{\alpha}_1\dot{\beta}_1 \sin\beta_0$. Substituting here $\dot{\alpha}_1$ and $\dot{\beta}_1$, we obtain the desired formula. In this case the drift of the gyroscope is analogous to the "nonholonomic error" considered by A. Yu. Ishlinskii ($^4$).

where $M'_Y$ is the constant component of the moment $M_Y$. Solving the general equations by the method of successive approximations gives

$$u' = -\frac{A_2 + C_1}{H\cos^2\beta_0}\left(\overline{\dot{\alpha}_1^2 \sin\beta_0} + \overline{\dot{\alpha}_1\dot{\theta}_1 \cos\beta_0}\right);$$

here the bar denotes averaging with respect to time. It is easy to show by simple calculations that the two expressions for $u'$ coincide. Solving the linear equations for given initial conditions, one can obtain an analytic formula for $u'$, but it is very cumbersome. Therefore we shall only point out that, as $k \to \infty$, $u'$ tends to the value given in no. 1, and that $u' = 0$ when $\sin\beta_0 = 0$.

Let us note one more interesting feature of the problem under consideration. If $\beta_0 = 0$, the system of equations of the first approximation degenerates, but the formula for $u'$ shows that drift exists when $\overline{\dot{\alpha}_1\dot{\theta}_1} = 0$. For example, if a moment

$$M = (M_x i_2 + M_z k_2)\cos pt,$$

is applied to the outer ring, then

$$(A_2 + A_1 + A)\ddot{\alpha}_1 + H\dot{\beta}_1 = M_x \cos pt, \qquad (A + B_1)\ddot{\beta}_1 - H\dot{\alpha}_1 = 0,$$

$$(C_2 + C_1)\ddot{\theta}_1 + k\theta_1 = M_z \cos pt;$$

denoting $\nu = H/\sqrt{(A_2 + A_1 + A)(A + B_1)}$, $\lambda = \sqrt{k/(C_2 + C_1)}$, we obtain

$$u' = -\frac{A_2 + C_1}{2H}\,\frac{p^2}{A^*C^*}\,\frac{M_{xM}z}{(p^2 - \nu^2)(p^2 - \lambda^2)};$$

here $A^* = A + A_1 + A_2$ and $C^* = C_1 + C_2$.

5°. In conclusion we make one more remark concerning the choice of the number of additional degrees of freedom in analyzing the influence of the elasticity of the suspension-axis supports on the nutational oscillations of a gyroscope. Let $R_{x_1}$, $R_{y_1}$ denote the moments of the reactions in the gyroscope supports acting from the inner ring, and $Q_{x_1}$ and $Q_{z_1}$ the moments of the reactions in the supports of the inner ring from the outer one; then

$$R_{x_1} = \frac{I_0 - A\cos^2\beta_0}{\cos\beta_0}\,\nu_0\Omega\sin\nu_0 t, \qquad R_{y_1} = -\frac{B_1}{\cos\beta_0}\sqrt{\frac{I_0}{A+B_1}}\,\nu_0\Omega\cos\nu_0 t,$$

$$Q_{x_1} = \frac{A_1 + C_1\cos^2\beta_0}{\cos\beta_0}\,\nu_0\Omega\sin\nu_0 t, \qquad Q_{z_1} = -C_1\sin\beta_0\,\nu_0\Omega\sin\nu_0 t,$$

$$P_{z_2} = -(A_2 + C_1)\operatorname{tg}\beta_0\,\nu_0\Omega\sin\nu_0 t, \qquad P_{y_2} = 0;$$

the subscript indicates the axis along which the moment of the reactions is directed.

Support reactions usually arise due to negligibly small elastic deformations. For a sufficiently large angular momentum, these deformations may become appreciable, since the reactions are proportional to the magnitude of the intrinsic rotation speed. It is then necessary to introduce additional degrees of freedom, the number of which is equal to the number of nonzero moments of support reactions. In the general case it is necessary to introduce 5 additional degrees of freedom. However, if some of the supports have very large stiffness, the number of additional degrees of freedom may be reduced.

If at the initial instant $\beta_0 = 0$ (the rings of the Cardan suspension are perpendicular), then the number of additional degrees of freedom is reduced to 3. Such a case may occur in determining the dynamic characteristics of a single-axis force gyrostabilizer with one gyroscope.

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*Note: Figure translations are in progress. See original paper for figures.*

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