MATHEMATICS

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Abstract

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ON ANALYTIC SOLUTIONS OF THE TRICOMI EQUATION

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§ 1. A real function $z(x, y)$ of real variables x, y is called a **regular solution of the Tricomi equation**

$$y\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \tag{1}$$

in a domain D if it is continuous in this domain together with its first partial derivatives and satisfies equation (1) there.

We shall call a regular solution $z(x, y)$ a **real analytic integral of equation (1) in a neighborhood of a point** (x_0, y_0) from D if the function $z(x, y)$ is analytic in some rectangle $|x - x_0| < a,\ |y - y_0| < b$, i.e. is expandable in a double power series

$$\sum_{n,m} b_{n,m}(x - x_0)^n (y - y_0)^m,$$

absolutely convergent inside this rectangle. It is clear that every real integral $z(x, y)$ of equation (1) is also defined for complex values of x, y. It is an analytic function in the bicylinder $|x - x_0| < a,\ |y - y_0| < b$, taking real values for real x and y. If the latter condition is dropped, then it is natural to call a complex-valued function $z(x, y)$, analytic in some bicylinder $|x - x_0| < r_1,\ |y - y_0| < r_2$ and satisfying the equation there, a **complex analytic** (or simply analytic) integral of equation (1) in a neighborhood of (x_0, y_0).

Every analytic integral $z(x, y)$ in a neighborhood of (x_0, y_0) can be represented in the form $z(x, y) = v_1(x, y) + iv_2(x, y)$, where v_1 and v_2 are real analytic integrals in a neighborhood of (x_0, y_0). If by R_1 we denote the class of analytic integrals, and by R_2 the class of real analytic integrals, then $R_1 \supset R_2$.

Let $z(x, y)$ be an analytic integral of equation (1) in a neighborhood of $(0, 0)$. Then the function $z(x, y)$ is analytic in some bicylinder $|x| < \rho_1,\ |y| < \rho_2$ and is represented as follows:

$$z(x,y)=\sum_{n=0}^{\infty}\frac{(-1)^n f_0^{(2n)}(x)}{(3n)!}\prod_{k=0}^{n-1}(3k{+}1)y^{3n}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1} f_1^{(2n)}(x)}{(3n+1)!}\prod_{k=0}^{n}(3k{-}1)y^{3n+1}; \tag{2}$$

here $f_0(x)$ and $f_1(x)$ are arbitrary functions analytic in the disk

$$|x|<\rho_1+\frac{2}{3}(\rho_2)^{3/2}.$$

Conversely, if the function $z(x,y)$ has the form (2) and the functions $f_0(x)$, $f_1(x)$ are analytic in the disk $|x|<\rho$, then $z(x,y)$ is a solution of equation (1), analytic in any bicylinder

$$|x|<R_1, \qquad |y|<\left[\frac{3}{2}(\rho-R_1)\right]^{2/3},$$

whatever $R_1<\rho$ may be. In particular, the general form of an integral entire in x,y (i.e. analytic in every bicylinder $|x|<R_1,\ |y|<R_2$)

of the Tricomi equation is given by formula (2), where $f_0(x)$ and $f_1(x)$ are arbitrary entire functions. Formula (2) also gives the general form of a real analytic integral in a neighborhood of $(0,0)$, if the functions $f_0(x)$ and $f_1(x)$ are regarded as functions analytic at $x=0$ with real Taylor coefficients.

In the present note we shall consider several problems for analytic solutions of the Tricomi equation that reduce to differential equations of infinite order.

§ 2. **Problem** T_1**.** Find a solution of equation (1), entire in x,y, satisfying, for all finite values of x, the conditions

$$u(x,0)=\varphi_0(x), \qquad u(x,c\sqrt[3]{x})=\varphi_1(x). \tag{3}$$

Here $\varphi_0(x)$, $\varphi_1(x)$ are given functions, and c is a given complex number. From representation (2) we find that $f_0(x)=\varphi_0(x)$, while the function $f_1(x)$ is determined from the differential equation of infinite order

$$f_1(x)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1} f_1^{(2n)}(x)}{(3n+1)!}\prod_{k=0}^{n}(3k-1)c\sqrt[3]{x}\,c^{3n}x^n =$$

$$=\varphi_1(x)-\sum_{n=0}^{\infty}\frac{(-1)^n\varphi_0^{(2n)}(x)}{(3n)!}\prod_{k=0}^{n-1}(3k+1)c^{3n}x^n. \tag{4}$$

It is easy to see that, for the solvability of problem T_1 in the class of integrals analytic in a neighborhood of $(0,0)$, it is necessary that the equality

$$\varphi_1(x) - \sum_{n=0}^{\infty} \frac{(-1)^n \varphi_0^{(2n)}(x)}{(3n)!} \prod_{k=0}^{n-1} (3k+1) c^{3n} x^n = \sqrt[3]{x}\, \nu(x),$$

hold, where $\nu(x)$ is a function analytic at the point $x = 0$. If this condition is satisfied, then equation (4) is rewritten as

$$f_1(x) + \sum_{n=1}^{\infty} \frac{(-1)^n f_1^{(2n)}(x)}{(3n+1)!} \prod_{k=1}^{n} (3k-1) c^{3n} x^n = \nu_1(x),$$

where $\nu_1(x) = \nu(x)/c$.

Investigating the last equation with the aid of work (1), we arrive at the following result:

Theorem 1. Problem T_1 is solvable if $\varphi_0(x)$ is an entire function of order $< 1/3$, and $\varphi_1(x)/\sqrt[3]{x}$ is an entire function of order $< 1/2$. Uniqueness holds in the class of solutions entire in x, y for which $z(x, 0)$ is an entire function of order $< 1/3$, and $\partial z/\partial y|_{y=0}$ is an entire function of order $< 1/2$.

Analogous results hold for the problem with conditions

$$\left.\frac{\partial z}{\partial y}\right|_{y=0} = \varphi_1(x), \qquad z(x, c\sqrt[3]{x}) = \varphi_0(x).$$

Remark. If c is a real number and the Taylor coefficients of the entire functions $\varphi_0(x)$ and $\varphi_1(x)$ are real, then the solution of problem T_1 will be a real integral entire in x, y.

§ 3. **Problem** T_2**.** Find a solution $z(x, y)$, analytic in a neighborhood of $(0, 0)$, of equation (1), for which

$$z(x, 0) = \varphi_0(x); \qquad z(x, c\sqrt[3]{x^2}) = \varphi_1(x) \quad \text{for } |x| \le h. \tag{5}$$

As in problem T_1, $\varphi_0(x)$, $\varphi_1(x)$ are given functions, and c is a complex number.

The solution of problem T_2 will be found if from conditions (5) we determine the functions $f_0(x)$ and $f_1(x)$. As before, $f_0(x) = \varphi_0(x)$, while $f_1(x)$ satisfies the equation

$$cx^{2/3} f_1(x) = \sum_{n=1}^{\infty} \frac{(-1)^n f_1^{(2n)}(x)}{(3n+1)!} \prod_{k=1}^{n} (3k-1) c^{3n+1} x^{2n+2/3} =$$

$$= \varphi_1(x) - \sum_{n=0}^{\infty} \frac{(-1)^n \varphi_0^{(2n)}(x)}{(3n)!} \prod_{k=1}^{n-1} (3k+1) c^{3n} x^{2n}. \tag{6}$$

For the existence of an integral of problem T_2 analytic in a neighborhood of $(0,0)$, it is necessary that the function

$$\nu(x) = c^{-1}x^{-2/3}\left[\varphi_1(x) - \sum_{n=0}^{\infty}\frac{(-1)^n f_0^{(2n)}(x)}{(3n)!}\prod_{k=0}^{n-1}(3k+1)c^{3n}x^{2n}\right]$$

be analytic at $x = 0$. If this circumstance holds, then equation (6) can be rewritten as follows:

$$f_1(x) + \sum_{n=1}^{\infty}\frac{(-1)^n f_1^{(2n)}(x)}{(3n+1)!}\prod_{k=1}^{n}(3k-1)c^{3n}x^{2n} = \nu(x). \tag{7}$$

Thus, in order to determine the sought function $f_1(x)$, we have an Euler differential equation of infinite order. Let

$$f_1(x) = \sum_{m=1}^{\infty} c_m x^m, \qquad \nu(x) = \sum_{m=0}^{\infty} g_m x^m.$$

If by a solution of equation (7) one understands a function $f_1(x)$ such that, upon substituting it into the equation, its left-hand side converges uniformly to $\nu(x)$ in some neighborhood of the origin of coordinates (and only such solutions are needed by us), then equation (7), in the class of these solutions, is equivalent to the system

$$c_m\beta_m = g_m, \qquad m = 0, 1, 2, \dots, \tag{8}$$

where

$$\beta_m = \sum_{n=0}^{m}\frac{\alpha_n m!}{(m-n)!}, \qquad \alpha_0 = 1, \qquad \alpha_{2n-1} = 0,$$

$$\alpha_{2n} = \frac{(-1)^n c^{3n}\prod_{k=1}^{n}(3k-1)}{(3n+1)!}, \qquad n = 1, 2, \dots; \quad m = 1, 0, \dots \tag{9}$$

Assume first that c is a real negative number. Then for the numbers β_m it is not difficult to obtain the estimate

$$\frac{d}{m^{3/2}}(1+\bar{x})^m \le |\beta_m| \le A(1+\bar{x})^m,$$

$$m = 1, 2, \dots, \qquad A < \infty, \qquad d = -c^3, \qquad \bar{x} = \sqrt[2/3]{d},$$

from which the estimate for the coefficients c_m also follows immediately. The final result can be formulated in the following form:

Theorem 2. Let $c < 0$, let the function $\varphi_0(x)$ be analytic in the disk $|x| < \rho$, and let $\varphi_1(x)$ have the form $\varphi_1(x) = \lambda(x) + x^{2/3}\mu(x)$, where $\lambda(x)$ and $\mu(x)$ are functions analytic in the disk

$$|x| < \frac{\rho}{1 + \sqrt[2/3]{|c|^3}}.$$

Then problem T_2 has a solution $z(x, y)$, analytic in the bicylinder

$$|x| < R, \qquad |y| < \left[\frac{3}{2}(\rho - R)\right]^{2/3}$$

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for any $R < \rho$. The condition $z(x, 0) = \varphi_0(x)$ is satisfied in the disk $|x| < \rho$, and the condition $z(x, c\sqrt[3]{x^2}) = \varphi_1(x)$ in the domain

$$|x| < \frac{\rho}{1 + \frac{2}{3}|c|^{3/2}}.$$

The solution of problem T_2 is unique in the class of all functions $v(x, y)$ analytic in a neighborhood of $(0, 0)$.

If the functions $\varphi_0(x)$ and $\varphi_1(x)$ have real Taylor coefficients, then the solution of problem T_2 will be a real analytic integral in a neighborhood of $(0, 0)$.

Let us now consider the case where c is an arbitrary complex number. From formulas (9) it is not difficult to obtain that

$$\lim_{m \to \infty} \sqrt[m]{|\beta_m|} \leq 1 + \frac{2}{3}|c|^{3/2}.$$

Moreover, the numbers $\beta_m/m!$ are the Taylor coefficients of the entire function

$$h(z) = e^z \omega(c^{3/2} z), \qquad \omega(z) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \prod_{k=1}^{n}(3k-1)}{(3n+1)!} x^{2n},$$

an exponential function of type 2/3.

Theorem 3. Suppose that the following assumptions are satisfied:

1) all Taylor coefficients $\beta_m/m!$ of the function $h(z)$ are nonzero, and

$$\lim_{m \to \infty} \sqrt[m]{|\beta_m|} = a > 0;$$

2) the function $\varphi_0(z)$ is analytic in the disk $|x| < R$, and

$$\varphi_1(x) = \lambda(x) + x^{2/3}\mu(x),$$

where $\lambda(x)$, $\mu(x)$ are functions analytic in the disk

$$|x| < \frac{R}{1 + \frac{2}{3}|c|^{3/2}}.$$

Then problem T_2 has a solution $z(x, y)$, unique in the class of functions analytic in a neighborhood of $(0, 0)$. The solution $z(x, y)$ is analytic in the bicylinder

$$|x| < r, \quad |y| < \left[\frac{3}{2}\left(\frac{aR}{1 + \frac{2}{3}|c|^{3/2}} - r\right)\right]^{2/3}$$

for any

$$r < \frac{aR}{1 + \frac{2}{3}|c|^{3/2}}.$$

In conclusion, we note that the requirement that the coefficients β_m not vanish is essential, since otherwise the uniqueness of the solution of the problem is violated. In particular, if c is a positive number, it can be shown that, if problem T_2 is solvable at all, then it necessarily has several solutions analytic in a neighborhood of $(0, 0)$.

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REFERENCES

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Note: Figure translations are in progress. See original paper for figures.

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