MATHEMATICS

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Abstract

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ON SMALL PERTURBATIONS OF AUTOMORPHISMS OF THE TORUS

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§ 1. Let the two-dimensional torus T^2 be realized as the unit square of the plane (x_1, x_2) with pairwise identified sides. An automorphism of the torus T^2 is a transformation $x \to Ax = \bar{x}$, defined by an integral matrix $A = \|a_{ij}\|$ with determinant ± 1, and acting on the torus by the formula

$$\bar{x}_i = \sum_j a_{ij} x_j \pmod 1, \quad i = 1, 2.$$

Suppose that the matrix $\|a_{ij}\|$ has two real eigenvalues, not equal in modulus to 1. Then, if $(\alpha_1, 1)$ is an eigenvector of the matrix A^* with eigenvalue λ_1, $|\lambda_1| < 1$, the system of straight lines on the torus

$$dx_2 + \alpha_1 dx_1 = 0 \tag{1}$$

has, with respect to the automorphism A, the following properties:

I. Every straight line Γ from the family (1) is carried under the action of A into a straight line $A\Gamma$ from the family (1), i.e. the family (1) is invariant with respect to A.

II. There exists μ_1, $\mu_1 > 1$, such that the lengths $s(l)$, $s(Al)$ of segments l, Al, lying on the straight lines Γ and $A\Gamma$, satisfy the inequality

$$s(Al) \geq \mu_1 s(l). \tag{2}$$

Similarly, if one takes the system of straight lines

$$dx_2 + \alpha_2 dx_1 = 0, \tag{1'}$$

where $(\alpha_2, 1)$ is an eigenvector of the transformation A^* with eigenvalue λ_2, $|\lambda_2| > 1$, then for it properties I and II′ are fulfilled, where II′ is formulated in the same way as II, only instead of inequality (2) there appears the inequality

$$s(Al) \leq \mu_2 s(l), \quad 0 < \mu_2 < 1. \tag{2'}$$

§ 2. It turns out that the property of the transformation A of possessing a family of curves with properties I, II (and I, II′) is coarse, i.e. is preserved under small perturbations of the automorphism A by nonlinear terms. Namely, let $A_\varepsilon = A + \varepsilon B(x)$, i.e. $x \to A_\varepsilon x = Ax + \varepsilon B(x)$, where $B(x) = (b_1(x_1, x_2), b_2(x_1, x_2))$, $b_i(x)$ are functions periodic in each argument with period 1 and three times continuously differentiable.

Theorem 1. *For sufficiently small $\varepsilon > 0$ there exists a system of curves*

$$dx_2 + \tilde{\alpha}_1(x, \varepsilon)\, dx_1 = 0, \tag{3}$$

possessing, with respect to the automorphism A_ε, properties I and II; the function $\tilde{\alpha}_1(x, \varepsilon)$ has continuous derivatives of bounded variation with respect to x, and is continuous with respect to ε. Among the solutions of (3) there are no closed curves.

Proof. We shall use the method of successive approximations. Suppose that the curves

$$dx_2 + \alpha_1^n(x, \varepsilon)\, dx_1 = 0$$

have already been constructed. Apply to them the transformation A_ε. If the matrix

$$\|a_{ij} + \varepsilon\, \partial b_i / \partial x_j\|^{-1}$$

has the form

$$\|\bar{a}_{ij}\| + \varepsilon \|g_{ij}(x, \varepsilon)\|,$$

where $\bar{a}_{ij}$ are the elements of the matrix A^{-1}, and $\|g_{ij}(x, \varepsilon)\|$ is a matrix depending continuously on x and on ε and bounded, then the result-

the system of curves will be written in the form $dx_2 + \alpha_1^{n+1}(x, \varepsilon)\, dx_1 = 0$, where

$$\alpha_1^{n+1}(x, \varepsilon) = \frac{(\bar{a}_{11} + \varepsilon g_{11}(x, \varepsilon))\alpha_1^n(A_\varepsilon^{-1}x) + (\bar{a}_{21} + \varepsilon g_{21}(x, \varepsilon))}{(\bar{a}_{12} + \varepsilon g_{12}(x, \varepsilon))\alpha_1^n(A_\varepsilon^{-1}x) + (\bar{a}_{22} + \varepsilon g_{22}(x, \varepsilon))}. \tag{4}$$

Lemma 1. *If $\max_x |\alpha_1^n(x, \varepsilon) - \alpha_1| < \delta$, then there exist μ, $0 < \mu < 1$, and $C < \infty$, depending only on A and B, such that*

$$\max_x |\alpha_1^{n+1}(x, \varepsilon) - \alpha_1| \le \mu\delta + C\varepsilon.$$

Proof. We rewrite equality (4) as follows:

$$\alpha_1^{n+1}(x, \varepsilon) = \frac{\bar{a}_{11}\alpha_1 + \bar{a}_{21} + (\alpha_1^n(A_\varepsilon^{-1}x) - \alpha_1)\bar{a}_{11} + \varepsilon(g_{11}\alpha_1^n + g_{21})}{\bar{a}_{12}\alpha_1 + \bar{a}_{22} + (\alpha_1^n(A_\varepsilon^{-1}x) - \alpha_1)\bar{a}_{12} + \varepsilon(g_{12}\alpha_1^n + g_{22})}.$$

From the fact that $(\alpha_1, 1)$ is an eigenvector of the matrix A^*, it is not hard to derive that $\bar{a}_{11}\alpha_1 + \bar{a}_{21} = \alpha_1/\lambda_1$, $\bar{a}_{12}\alpha_1 + \bar{a}_{22} = 1/\lambda_1$. Therefore

$$\alpha_1^{n+1}(x, \varepsilon) = \frac{\alpha_1 + \lambda_1[(\alpha_1^n(A_\varepsilon^{-1}x) - \alpha_1)\bar{a}_{11} + \varepsilon(g_{11}\alpha_1^n + g_{21})]}{1 + \lambda_1[(\alpha_1^n(A_\varepsilon^{-1}x) - \alpha_1)\bar{a}_{12} + \varepsilon(g_{12}\alpha_1^n + g_{22})]},$$

$$|\alpha_1^{n+1}(x,\varepsilon)-\alpha_1| = \left|\lambda_1 \frac{(\alpha_1-\alpha_1^n)(-\bar{a}_{12}\alpha_1+\bar{a}_{11})+\varepsilon[g_{11}\alpha_1^n+g_{21}-\alpha_1(g_{12}\alpha_1^n+g_{22})]}{1+\lambda_1[(\alpha_1^n(A_\varepsilon^{-1}x)-\alpha_1)\bar{a}_{12}+\varepsilon(g_{12}\alpha_1^n+g_{22})]}\right|.$$

To complete the proof of the lemma it is enough to take into account that, analogously to the preceding, $-\alpha_1\bar{a}_{12}+\bar{a}_{11}=\lambda_1$.

It follows from Lemma 1 that for any $\delta > 0$ one can find $\varepsilon_1(\delta)$ such that $\max_x |\alpha_1^n(x,\varepsilon)-\alpha_1| < \delta$ for all n, if $\varepsilon < \varepsilon_1(\delta)$.

We now estimate the difference $|\alpha_1^{n+1}(x,\varepsilon)-\alpha_1^n(x,\varepsilon)|$. Using (4), it is easy to find that

$$\alpha_1^{n+1}(x,\varepsilon)-\alpha_1^n(x,\varepsilon) = D_n(x,\varepsilon)[\alpha_1^n(A_\varepsilon^{-1}x,\varepsilon)-\alpha_1^{n-1}(A_\varepsilon^{-1}x,\varepsilon)],$$

where

$$D_n(x,\varepsilon) = \frac{\bar{a}_{11}\bar{a}_{22}-\bar{a}_{12}\bar{a}_{21}+\varepsilon^2(g_{11}g_{22}-g_{12}g_{21})+\varepsilon(g_{11}\bar{a}_{22}+g_{22}\bar{a}_{11}-g_{21}\bar{a}_{12}-g_{12}\bar{a}_{21})}{[(\bar{a}_{12}\alpha_1^n+\bar{a}_{22})+\varepsilon(g_{12}\alpha_1^n+g_{22})][(\bar{a}_{12}\alpha_1^{n-1}+\bar{a}_{22})+\varepsilon(g_{12}\alpha_1^{n-1}+g_{22})]}.$$

But, by Lemma 1, for any $\delta > 0$ and $\varepsilon < \varepsilon_2(\delta)$ for all n

$$|\bar{a}_{12}\alpha_1^n+\bar{a}_{22}-1/\lambda_1| \le \delta.$$

Since $\bar{a}_{11}\bar{a}_{22}-\bar{a}_{12}\bar{a}_{21} = \pm 1$ and $|\lambda_1| < 1$, there exists ρ, $0<\rho<1$, such that for all n

$$\max_x |\alpha_1^{n+1}(x,\varepsilon)-\alpha_1^n(x,\varepsilon)| \le \rho \max_x |\alpha_1^n(x,\varepsilon)-\alpha_1^{n-1}(x,\varepsilon)|.$$

Consequently, the sequence $\alpha_1^n(x,\varepsilon)$ converges uniformly on T^2. Put

$$\tilde{\alpha}(x,\varepsilon) = \lim_{n\to\infty} \alpha_1^n(x,\varepsilon).$$

It is easy to see that the system of curves Γ_1

$$dx_2 + \bar{\alpha}_1(x,\varepsilon)\,dx_1 = 0 \qquad (5)$$

has properties I and II with respect to the automorphism A_ε. The continuous dependence of $\tilde{\alpha}_1(x,\varepsilon)$ on ε is obvious. The existence of derivatives of $\tilde{\alpha}_1(x,\varepsilon)$ with respect to x of bounded variation was established by V. I. Oseledets by methods analogous to the preceding ones.

We now prove that among the curves (5) there are no closed curves. Suppose that such a curve Γ exists. Then, by property II, the length of the curve $A_\varepsilon^{-n}\Gamma$

satisfies the inequality $s(A_\varepsilon^{-n}\Gamma) \leq \mu_1^{-n} s(\Gamma)$. But the curve $A_\varepsilon^{-n}\Gamma$ cannot then be a curve of (5), since it cannot satisfy

to the inequality $|\tilde{a}_1 - a_1| < \delta$, which holds for the curves (5). The theorem is proved.

Remark. Similarly one can prove the existence of curves Γ_2, $dx_2 + \tilde{a}_2(x, \varepsilon)\, dx_1 = 0$, possessing properties I and II′.

§ 3. Consider two (incompatible) equations on the torus

$$dx_2 = f_i(x_1, x_2)\, dx_1 \quad (i = 1, 2), \tag{6}$$

where the first derivatives of the functions $f_i(x_1, x_2)$ of period 1 in x_1 and x_2 have bounded variation. Poincaré ([1]) defined rotation numbers ω_i for equations of the form (6).

Theorem 2. *Let ω_1, ω_2 be irrational and*

$$-\infty < c_1 < f_1 < C_1 < c_2 < f_2 < C_2 < \infty, \tag{7}$$

where $c_1, \dots, C_2$ are constants. Then there exists a homeomorphism of the torus $x \leftrightarrow y$, straightening the integral curves Γ_i of both equations (6), i.e. transforming them into straight lines Γ'_i: $dy_2 = \omega_i dy_1$ $(i = 1, 2)$.

Proof. 1°. Denote by $\Gamma_i(x_0)$ the integral curves of equations (6) passing through the point $x_0 = (x_1^0, x_2^0)$, and by $\Gamma'_i(y_0)$ the straight line $y_2 - y_2^0 = \omega_i(y_1 - y_1^0)$. Let p_1, p_2 be two integer points. Consider on the x-plane the point $q(p_1, p_2) = \Gamma_1(p_1) \cap \Gamma_2(p_2)$, and on the y-plane the point $q'(p_1, p_2) = \Gamma'_1(p_1) \cap \Gamma'_2(p_2)$. Define the mapping $y \to x$ by assigning to the point $q'(p_1, p_2)$ the point $q(p_1, p_2)$.

2°. **Lemma 2.** *The mapping $q' \to q$ is uniformly continuous.*

Proof. By Denjoy' s theorem ([2]), there exists a continuous transformation of the torus taking the lines Γ_1 into the straight lines Γ'_1. Therefore, for every $\varepsilon > 0$ there exists $\delta_1(\varepsilon) > 0$ such that if the distance between two straight lines $\Gamma'_1(p_1), \Gamma'_1(p_3)$ is less than δ_1, then the distance between the curves $\Gamma_1(p_1), \Gamma_1(p_3)$ is everywhere less than ε. Analogous arguments applied to Γ'_2 and Γ_2 give $\delta_2(\varepsilon)$. In view of condition (7), if the distance between the points $q'(p_1, p_2)$, $q'(p_3, p_4)$ is less than $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$, then the distance between the points $q(p_1, p_2)$, $q(p_3, p_4)$ is less than $K\varepsilon$ (where K depends only on C_1 and c_2). The lemma is proved.

3°. Since the sets of points $q(p_1, p_2)$ and $q'(p_1, p_2)$ are everywhere dense, in view of ([2]) and (7), the mapping $q' \to q$ can be extended by continuity to the whole y-plane. The resulting homeomorphism of the plane determines the required homeomorphism of the torus, since the straight lines Γ'_i are mapped into the curves Γ_i. Theorem 2 is proved.

§ 4. **Theorem 3.** *An ergodic automorphism of the two-dimensional torus is structurally stable *. This means that, under the conditions of Theorem 1 and for sufficiently small ε, there exists a homeomorphism of the torus $x \leftrightarrow y$, transforming the perturbed automorphism into the unperturbed one:*

$$y(A_\varepsilon x) = Ay(x). \tag{8}$$

Proof. For small ε, A_ε has one fixed point O_ε on the x-plane. We take it as the origin on the x-plane. Construct the homeomorphism of Theorem 2 from the curves Γ_i obtained in § 2 (5). Since these curves on the torus are not closed, the rotation numbers ω_i are irrational. On the other hand, $\omega_i(\varepsilon)$ depend continuously on ε. Consequently, they are constant. Therefore the straight lines Γ_i' have the directions of the eigenvectors of A.

The curves $\Gamma_i(O_\varepsilon)$ are mapped into themselves under the action of the transformation A_ε, as are the straight lines $\Gamma_i'(O)$ under the action of A. Moreover, A takes $q'(p_1, p_2)$ to $q'(Ap_1, Ap_2)$, while A_ε takes $q(O_\varepsilon + p_1, O_\varepsilon + p_2)$ to $q(O_\varepsilon + Ap_1, O_\varepsilon + Ap_2)$. Therefore (8) is fulfilled for $x = q(p_1, p_2)$, and hence, by continuity, for all x.

Remark. If A_ε is analytic, then, according to ([3]), the curves Γ_1 or

* Or "rough," in the terminology of Andronov–Pontryagin.

Γ_2 can be straightened locally by an analytic transformation. However, the homeomorphism constructed in § 3 may fail to be differentiable. Indeed, the eigenvalues of A_ε in O_ε may differ from the eigenvalues of A. If the homeomorphism $x \leftrightarrow y$ is absolutely continuous, then, for sufficiently small ε, the transformation A_ε has an invariant measure absolutely continuous with respect to Lebesgue measure and is metrically isomorphic to A. However, we do not know whether the condition of absolute continuity is fulfilled even in the case of analytic measure-preserving perturbations.

§ 5. In the n-dimensional case we have been able to prove the following.

Theorem 4. *If the matrix $A = \|a_{ij}\|$ has n real eigenvalues, k of which are greater than 1 and the remaining ones less than 1, and $A_\varepsilon = A + \varepsilon B$, then on the torus T^n there exists a system of $(n-k)$-dimensional unclosed smooth surfaces, invariant with respect to A_ε, and such that for any piece l of a surface the volume $V(l) \le \mu^l V(A_\varepsilon l)$, where μ'; $0 < \mu' < 1$.*

Theorem 5. *Let the n-dimensional torus T^n be decomposed into $(n-1)$-dimensional smooth surfaces $\Gamma : x_n = g(x_1, \dots, x_{n-1})$, and let the functions g be twice continuously differentiable. If none of these surfaces is closed, then there exists a homeomorphism of the torus $x \leftrightarrow y$ straightening the surfaces Γ, i.e. taking them into the planes $\Gamma' : y_n = \omega_1 y_1 + \dots + \omega_{n-1} y_{n-1} + C$.*

Proof. 1°. Let $\Gamma(x_n)$ be the surface Γ passing through the point $(0, \dots, 0, x_n)$. Denote by $Q_p(x_n)$ (where $p = (p_1, \dots, p_{n-1})$) the point $(p, x'_n) \in \Gamma(x_n)$. Let $p_1, \dots, p_{n-1}$ be integers. Then Q_p may be regarded as a mapping of the circle $p = 0$ onto itself. It is easy to see that $Q_{p+q} = Q_{pQ}q$, so that all the mappings Q_p commute.*

2°. **Lemma 3.** *Suppose a finite number of commuting twice continuously differentiable homeomorphisms of the circle $Q_1, \dots, Q_r$ is given. Then either there exists a homeomorphism of the circle transforming them into rotations, or there exists an N such that $Q_1^N, \dots, Q_r^N$ have a common fixed point.*

Indeed, if at least one of the transformations, for example Q_1, has an irrational rotation number ([1]), then, for a certain choice of parameter on the circle, it is a rotation through an irrational angle. Since all the transformations Q commute with Q_1^k $(k = 1, 2, \dots)$, they commute with all rotations and, consequently, themselves are rotations. If all rotation numbers are rational, then for some N each of the transformations $R_1 = Q_1^N, \dots, R_r = Q_r^N$ has fixed points. Therefore there exists

$$\lim_{n_1 \to \infty} R_1^{n_1} \lim_{n_2 \to \infty} R_2^{n_2} \cdots \lim_{n_r \to \infty} R_r^{n_r} x_0 = z.$$

It is easy to see that z is a fixed point of all $R_1, \dots, R_r$.

3°. Apply Lemma 3 to the transformations $Q_i = Q_{1,0,\dots,0}, \dots, Q_{0,\dots,0,1}$. If all Q^N have a common fixed point z, then the surface $\Gamma(z)$ is closed, contrary to the hypothesis of the theorem. Consequently, there exists a parameter $\varphi(x_n)$ such that $\varphi(Q_i x_n) = \varphi(x_n) + \omega_i$. Now define $x_0(x)$ by the condition $x \in \Gamma(x_0)$ and set $y_i = x_i$ $(1 \leq i < n)$, $y_n = \varphi(x_0(x)) + \omega_1 x_1 + \dots + \omega_{n-1} x_{n-1}$. It is easy to see that $x \leftrightarrow y$ is the required homeomorphism of the torus.

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* The same device makes it possible to study decompositions of skew products with circle fiber. The question reduces to finding representations of the fundamental group of the base into the group of mappings of the circle onto itself.