MATHEMATICS

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Abstract

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ON FUNCTIONS SATISFYING THE DIFFERENTIAL EQUATION $x^2y''' + 3xy'' + y' + x^2y = 0$

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In the present note, by an operational method we solve the linear ordinary differential equation with variable coefficients

$$x^2y''' + 3xy'' + y' + x^2y = 0. \tag{1}$$

After the change of variable by the formula $x = 3\sqrt[3]{t}$, equation (1) is reduced to the form

$$\frac{d}{dt}\,t\,\frac{d}{dt}\,t\,\frac{d}{dt}\,y + y = 0. \tag{2}$$

This equation is of importance in constructing an operational calculus for the operator

$$T = \frac{d}{dt}\,t\,\frac{d}{dt}\,t\,\frac{d}{dt}$$

(^{1}).

To obtain three linearly independent solutions of the latter equation by the operational method, it is expedient to consider the equation

$$t^3y''' + 3t^2y'' + ty' + ty = 0, \tag{3}$$

which is obtained from (2) by multiplying by t.

Let

$$\varphi(p) = \int_0^\infty e^{-pt}y(t)\,dt.$$

Then equation (3) is reduced to the operational equation

$$p^3\varphi''' + 6p^2\varphi'' + (7p+1)\varphi' + \varphi = 0. \tag{4}$$

After the change of variable by the formula $z = \frac{1}{p}$, equation (4) becomes

$$z^3\varphi''' + z(z+1)\varphi' - \varphi = 0. \tag{5}$$

Equation (5) is a special case of an equation depending on a parameter ν:

$$z^3\varphi''' + \left[z(z+1) - \left(\frac{\nu}{2}\right)^2 z\right]\varphi' - \left[1 - \left(\frac{\nu}{2}\right)\right]\varphi = 0, \tag{6}$$

when this parameter is equal to zero. If the parameter ν is not an integer, then the functions

$$\widetilde{\varphi}_1(z,\nu) = zJ_\nu(2\sqrt{z}); \tag{7}$$

$$\widetilde{\varphi}_2(z,\nu) = zY_\nu(2\sqrt{z}); \tag{8}$$

$$\widetilde{\varphi}_3(z,\nu) = z\Pi_\nu(2\sqrt{z}), \tag{9}$$

where

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\nu+2m}}{m!\Gamma(\nu+m+1)}, \qquad Y_\nu(z) = \frac{\cos(\pi\nu)J_\nu(z) - J_{-\nu}(z)}{\sin(\pi\nu)}$$

are Bessel functions,

$$\Pi_\nu(z) = \cos\left(\frac{\pi\nu}{2}\right) \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{\Gamma(m+1+\nu/2)\Gamma(m+1-\nu/2)}$$

is a Poisson function, form a fundamental system of solutions of equation (6). For $\nu = 0$ this no longer holds, since as $\nu \to 0$ the Poisson function $\Pi_\nu(z)$ degenerates into the Bessel function $J_\nu(z)$, and the triple of solutions (7), (8), (9) of equation (6) degenerates into a pair of linearly independent solutions of equation (5)

$$\varphi_1(z) = \lim_{\nu\to 0} \widetilde{\varphi}_1(z,\nu) = \lim_{\nu\to 0} \widetilde{\varphi}_3(z,\nu) = zJ_0(2\sqrt{z}); \tag{10}$$

$$\varphi_2(z) = \lim_{\nu\to 0} \widetilde{\varphi}_2(z,\nu) = zY_0(2\sqrt{z}). \tag{11}$$

To find the third linearly independent solution of equation (5), let us consider

$$\lim_{\nu\to 0}\frac{\widetilde{\varphi}_1(z,\nu)-\widetilde{\varphi}_3(z,\nu)}{\nu}=\frac{1}{2}\varphi_2(z).$$

We find

$$\lim_{\nu\to 0}\frac{\dfrac{d}{d\nu}\{\widetilde{\varphi}_1(z,\nu)-\widetilde{\varphi}_3(z,\nu)\}-\dfrac{1}{2}\varphi_2(z)}{\nu}=\frac{\pi^2}{4}\varphi_1(z)+\varphi_3(z),$$

where

$$\varphi_3(z)=\frac{1}{4}\sum_{m=0}^{\infty}\frac{(-1)^m z^{m+1}}{(m!)^2}\{\ln^2 z-4\psi(m+1)\ln z+4\psi^2(m+1)-2\psi'(m+1)\},$$

$$\psi(m+1)=\frac{d}{dm}\ln\Gamma(m+1).$$

The function $\varphi_3(z)$ satisfies equation (5) and is linearly independent of $\varphi_1(z)$ and $\varphi_2(z)$. After passing from the images $\varphi_1\left(\frac{1}{p}\right)$, $\varphi_2\left(\frac{1}{p}\right)$, $\varphi_3\left(\frac{1}{p}\right)$ to the originals, we obtain, respectively, three linearly independent solutions of equation (3):

$$y_1(t)=\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^3}t^m=J_{0,0}^{(2)}(3\sqrt[3]{t}),$$

$$y_2(t)=J_{0,0}^{(2)}(3\sqrt[3]{t})\ln t-3\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^3}t^m\psi(m+1),$$

$$y_3(t)=\frac{1}{4}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^3}t^m\{\ln^2 t-6\psi(m+1)\ln t+9\psi^2(m+1)-3\psi'(m+1)\}.$$

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1. A. P. Prudnikov, DAN, **142**, No. 4 (1962).

Note: Figure translations are in progress. See original paper for figures.

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