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# MATHEMATICS

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1962

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**Abstract**

**Full Text**

MATHEMATICS

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## ON ESTIMATES OF THE CHROMATIC NUMBER OF CONNECTED GRAPHS

*(Presented by Academician A. I. Mal' tsev on 11 VIII 1961)*

In this paper the following notions are assumed to be known: **graph**, **subgraph**, **vertex** of a graph, **edge** of a graph, **connectedness** of a graph, **adjacency** of vertices, **path** between two vertices <sup>(1)</sup>.

In graph theory the following operation on graphs is considered: all vertices of a given graph  $G$  are partitioned into groups in such a way that no two vertices assigned to the same group (containing more than one vertex) are adjacent. Historically this operation received the name of **coloring** (of the vertices) of a graph; in coloring it is assumed that vertices assigned to one group are colored with one color. The result of the coloring operation is called a **coloring** of the graph  $G$ ; the number of resulting groups determines the **number of colors** required for the given coloring. Usually one is interested in **minimal colorings**, i.e., those for which the required number of colors does not exceed the number of colors required for any coloring of the graph  $G$ . The number of colors in a minimal coloring is called the **chromatic number**  $h(G)$  of the graph  $G$ . The subject of the present paper will be the study of the behavior of the chromatic number of a connected graph as a function of the number of its vertices  $n$  and the number of its edges  $p$ .

Consider the class  $\mathcal{G}(n, p)$  of all connected graphs with  $n$  vertices and  $p$  edges, having neither loops nor parallel edges. In this case  $p$  satisfies the inequality

$$n - 1 \leq p \leq \frac{n(n-1)}{2}. \quad (1)$$

A graph with  $n$  vertices and  $\frac{n(n-1)}{2}$  edges is called  **$n$ -complete**. Let us note that the chromatic number of an  $n$ -complete graph is equal to  $n$ .

**Definition 1.** The **upper chromatic number**  $X(n, p)$  (the **lower chromatic number**  $\chi(n, p)$ ) of the class  $\mathcal{G}(n, p)$  is a chromatic number of some graph from  $\mathcal{G}(n, p)$  which is not less (not greater) than the chromatic number of any graph from  $\mathcal{G}(n, p)$ .

In the formulation of the theorem below and further on, brackets  $[ ]$  and  $\{ \}$  denote respectively the integral and fractional parts.

**Theorem.**

$$X(n, p) = \left\lceil \frac{3 + \sqrt{9 + 8(p - n)}}{2} \right\rceil; \quad (2)$$

$$\chi(n, p) = - \left\lceil - \frac{n}{\left\lfloor \frac{n^2 - 2p}{n} \right\rfloor} \left( 1 - \frac{\left\{ \frac{n^2 - 2p}{n} \right\}}{1 + \left\lfloor \frac{n^2 - 2p}{n} \right\rfloor} \right) \right\rceil. \quad (3)$$

**Definition 2.** The **length of a path** in a graph is the number of edges forming the given path. The **distance** between vertices

the graph is called the minimal length of a path joining  $Q$  and  $R$ . Two vertices  $Q$  and  $R$  in a graph  $G$  that are at distance 2 from one another are called **cocolored** if there exists a minimal coloring of the graph  $G$  in which  $Q$  and  $R$  are colored with the same color.

**Lemma 1.** *In an incomplete graph there exists a pair of cocolored vertices.*

**Proof of the lemma.** Let  $G$  be an incomplete graph. In this case it has at least one vertex  $Q$  having a nonempty set  $P_2$  of vertices lying at distance 2 from  $Q$ . Consider some minimal coloring of the graph  $G$ . Suppose that under this coloring the vertex  $Q$  is colored with color  $a$ . Two alternatives, A and B, are possible.

A. There is a vertex in  $P_2$  colored with color  $a$ . In this case the lemma is proved.

B. No vertex in  $P_2$  is colored with color  $a$ . In this case take any vertex  $R$  from  $P_2$ . Let it be colored with color  $b$ . Consider the set  $P_1$  of vertices adjacent to  $Q$ . Two cases, B1 and B2, are possible.

B1. No vertex in  $P_1$  is colored with color  $b$ . In this case we recolor the vertex  $Q$  with color  $b$ , which proves the cocoloredness of  $Q$  and  $R$ .

B2. There is a nonempty set  $P'_1$  of vertices from  $P_1$  colored with color  $b$ . Note that any vertex  $P$  from  $P'_1$  has no adjacent vertices colored with color  $a$ , except for the vertex  $Q$ . Indeed, any vertex adjacent to  $P$  will either be  $Q$ , or adjacent to  $Q$ , or will be at distance 2 from  $Q$ . But then, either because of adjacency to  $Q$ , or by assumption B, it cannot be colored with color  $a$ . “Erase” the color  $a$  from the vertex  $Q$ . In this case all vertices from  $P'_1$  can be recolored with color  $a$ . After this recoloring, the vertex  $Q$  can be colored with color  $b$ , which also proves the cocoloredness of  $Q$  and  $R$ .

We shall call the **identification** of vertices  $Q$  and  $R$  in a graph  $G$  the following transformation of the graph  $G$ : the vertices  $Q$  and  $R$  are replaced by one vertex, which is joined by edges to those and only those vertices of the graph  $G$  that were adjacent to at least one of the vertices  $Q$  and  $R$ .

**Lemma 2.** *Let the graph  $G'$  be obtained from the graph  $G$  by identifying a pair of cocolored vertices. Then  $h(G) = h(G')$ .*

The proof of Lemma 2 is elementary.

**Proof of (2).** Let a graph  $G$  from  $\mathcal{G}(n, p)$  be given. Consider the following procedure for determining  $h(G)$ . If the graph  $G$  is complete, then  $h(G) = h$ . If, however, the graph  $G$  is incomplete, then, by Lemma 1, it contains a pair of cocolored vertices. Identifying them, we obtain a new graph  $G_1$ , in which there will be one vertex fewer and at least one edge fewer than in the graph  $G$ . At the same time  $h(G_1) = h(G)$  by Lemma 2. Obtaining the graph  $G_1$  will be the first step. The same reasoning can be applied to the graph  $G_1$  as to the graph  $G$ , and so on. Finally, after the  $s$ -th ( $s < n$ ) step we obtain an  $(n - s)$ -complete graph  $G_s$ , for which

$$h(G_s) = h(G) = n - s. \quad (4)$$

The graph  $G_s$  will have  $h(G)(h(G) - 1)/2$  edges, i.e. it will have  $p - h(G)(h(G) - 1)/2$  fewer edges than the graph  $G$ . But since at each  $i$ -th step ( $i = 1, \dots, s$ ) the graph  $G_i$  contained at least one edge fewer than the graph  $G_{i-1}$  ( $G_0 = G$ ), it follows that

$$p - \frac{h(G)(h(G) - 1)}{2} \geq s,$$

or, by (4),

$$p - \frac{h(G)(h(G) - 1)}{2} \geq n - h(G). \quad (5)$$

Solving this inequality with respect to  $h(G)$ , we obtain

$$h(G) \leq \frac{3 + \sqrt{9 + 8(p - n)}}{2}. \quad (6)$$

for any graph  $G$  from  $\mathcal{G}(n, p)$ . We now find a graph  $G_B$  from  $\mathcal{G}(n, p)$  for which  $h(G_B) = \kappa$ , where

$$\kappa = \left\lceil \frac{3 + \sqrt{9 + 8(p - n)}}{2} \right\rceil. \quad (7)$$

From (7) and (1) it is easy to show that

$$\kappa \leq n, \quad \frac{\kappa(\kappa - 1)}{2} \leq p, \quad n - \kappa \leq p - \frac{\kappa(\kappa - 1)}{2}. \quad (8)$$

The graph  $G_B$  is constructed as follows: first a  $\kappa$ -complete graph is constructed; to any one of its vertices a chain containing  $n - \kappa$  vertices and  $n - \kappa$  edges is

attached; and the remaining  $p - \kappa(\kappa - 1)/2 - n + \kappa$  edges are placed arbitrarily. By virtue of (8), all these constructions are possible. Since  $G_B$  contains a  $\kappa$ -complete subgraph,  $h(G_B) \geq \kappa$ , whence, by virtue of (6), (2) follows.

**Proof of (3).** Consider any graph  $G$  from  $\mathcal{G}(n, p)$ . Let  $h(G) = k$ . We shall call an **absent edge**  $(Q, R)$  in the graph  $G$  any pair of nonadjacent vertices  $Q$  and  $R$ , as well as all pairs of the form  $(Q, Q)$ . Obviously, the number of absent edges in  $G$  is equal to  $n^2 - 2p$ . Consider any minimal coloring of the graph  $G$ . Let  $M_i$  be the set of vertices colored with the  $i$ -th color ( $i = 1, \dots, k$ ), and let it contain  $x_i$  vertices. Since all vertices in  $M_i$  are pairwise nonadjacent, the total number of absent edges formed only by vertices from  $M_i$  is equal to  $x_i^2$ . Hence the inequality follows immediately:

$$\sum_{i=1}^k x_i^2 \leq n^2 - 2p \quad (9)$$

under the condition that

$$\sum_{i=1}^k x_i = n. \quad (10)$$

Let us find the minimum  $m$  of the function  $F = \sum_{i=1}^k x_i^2$  under condition (10) on the set of integers. Put

$$x_i = \left\lfloor \frac{n}{k} \right\rfloor + y_i, \quad q = n - \left\lfloor \frac{n}{k} \right\rfloor k, \quad (11)$$

where  $n/k$  are the values of  $x_i$  minimizing  $F$  under condition (10) on the set of rational numbers. Obviously,

$$\sum_{i=1}^k y_i = q \quad (0 \leq q < k). \quad (12)$$

It is easy to show that finding the minimum of  $F$  reduces to finding the minimum of  $\sum_{i=1}^k y_i^2$  under condition (12). This minimum is attained, for example, when  $y_1 = \dots = y_{k-q} = 0$ ,  $y_{k-q+1} = \dots = y_k = 1$ . Hence it follows that

$$m = \frac{n^2}{k} + k \left( \frac{n}{k} - \left\lfloor \frac{n}{k} \right\rfloor \right) \left( 1 + \left\lfloor \frac{n}{k} \right\rfloor - \frac{n}{k} \right), \quad (13)$$

where  $n^2/k$  is the minimum of  $F$  on the set of rational numbers. Now (9) can be rewritten in the form

$$k^2 \left( \frac{n}{k} - \left[ \frac{n}{k} \right] \right) \left( 1 + \left[ \frac{n}{k} \right] - \frac{n}{k} \right) \leq (n^2 - 2p)k - n^2. \quad (14)$$

From this inequality it follows that

$$h(G) \geq \nu = -[-x_0] \quad (15)$$

for any  $G$  from  $\mathcal{G}(n, p)$ , where  $x_0$  is the root of the equation

$$x^2 \left( \frac{n}{x} - \left[ \frac{n}{x} \right] \right) \left( 1 + \left[ \frac{n}{x} \right] - \frac{n}{x} \right) = (n^2 - 2p)x - n^2, \quad (16)$$

lying in the interval from 1 to  $n$ .

Examining the behavior of the left- and right-hand sides of equation (16), it is easy to show that  $[n/x_0] = [n/x_1]$ , where  $x_1$  is the zero of the equation  $(n^2 - 2p)x - n^2 = 0$ . Computing  $[n/x_1]$  and substituting it into (16), we obtain a linear equation in  $x$ , from which we find  $x_0$ :

$$x_0 = \frac{n}{\left[ \frac{n^2 - 2p}{n} \right]} \left( \frac{\left\{ \frac{n^2 - 2p}{n} \right\}}{1 + \left[ \frac{n^2 - 2p}{n} \right]} \right). \quad (17)$$

We shall now construct a graph  $G_H$  from  $\mathcal{G}(n, p)$  for which  $h(G_H) = \nu$ . Since  $\nu \geq x_0$ ,  $\nu$  satisfies inequality (14) if in it  $k$  is replaced by  $\nu$ . Solving the resulting inequality with respect to  $p$ , we obtain that

$$p \leq p_1 = \left\lfloor \frac{1}{2} \left( n^2 - \frac{n^2}{\nu} - \nu \left( \frac{n}{\nu} - \left[ \frac{n}{\nu} \right] \right) \right) \left( 1 + \left[ \frac{n}{\nu} \right] - \frac{n}{\nu} \right) \right\rfloor. \quad (18)$$

At the same time, from (5) and (15) it follows that

$$p \geq p_2 = \frac{\nu(\nu - 1)}{2} + n - \nu. \quad (19)$$

Let us construct an auxiliary graph  $G^*$ . For this, distribute the  $n$  vertices into  $\nu$  groups  $M_i$ , with  $x_i$  vertices in each ( $i = 1, \dots, \nu$ ), so that  $\sum_{i=1}^{\nu} x_i^2$  is minimal. Join each vertex of  $M_i$  by edges to all vertices not belonging to  $M_i$ . Clearly, the total number of edges in the graph  $G^*$  will be equal to  $p_1$ . It is easy to show that  $G^*$  contains a  $\nu$ -complete subgraph  $G_\nu$ , and moreover one such that every vertex of  $G^*$  not belonging to  $G_\nu$  is adjacent to some vertex of  $G_\nu$ . In this connection, in  $G^*$  one can select a connected subgraph  $G'$  with  $n$  vertices and  $p$  edges. Now remove from  $G^*$  any  $p_1 - p$  edges that do not belong to the subgraph  $G'$ . In view of (18) and (19), all the constructions described are possible. As a

result we obtain a graph  $G_H$  from  $\mathcal{G}(n, p)$ . Since  $G^*$  is colorable with  $\nu$  colors, and  $G_H$  is obtained from  $G^*$  by deleting some edges,  $G_H$  is also colorable with  $\nu$  colors. Hence it follows that  $h(G) \leq \nu$ , whence, by (15), we obtain (3).

The authors express their gratitude to Yu. M. Voloshin for his attention to the work and for a number of useful comments.

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Received  
28 VIII 1961

## REFERENCES

G. König, *Theorie der endlichen und unendlichen Graphen*, Leipzig, 1936.

*Note: Figure translations are in progress. See original paper for figures.*

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