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Abstract

Full Text

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On Minimal Systems and Quasi-Complements in Banach Space

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Let E be a separable Banach space, and let E^* be its conjugate. A collection $\{x_i\}_1^\infty$ of elements of the space E is called minimal if each x_i does not belong to the closure of the linear span of the remaining elements. A collection of linear functionals $\{f_i\}_1^\infty$ will be called conjugate to $\{x_i\}_1^\infty$ if

$$f_i(x_j) = \delta_{ij} \quad (x_i \in E,\ f_i \in E^*;\ i,\ j = 1,\ 2, ...).$$

Every minimal system has a conjugate one. A set $G \subset E^*$ is called total with respect to a subspace $E_1 \subset E$ if, for any element $x \in E_1$, from the condition $g(x) = 0$ for all $g \in G$ it follows that $x = \theta$. We shall agree to call a subspace $E_1 \subset E$ nontrivial if both it itself and the quotient space E/E_1 are infinite-dimensional; in what follows only nontrivial subspaces are considered.

Theorem 1. *Suppose that in a subspace $E_1 \subset E$ a complete minimal system $\{x_i\}_1^\infty$ is defined, and that its conjugate system $\{f_i\}_1^\infty$ $(f_i \in E^*)$ is total with respect to E_1. Then the system $\{x_i\}_1^\infty$ can be extended to a minimal system, complete in E and having a total (with respect to E) conjugate system.*

Proof. In the quotient space E/E_1 (as in every separable Banach space (^{1})) there exists a complete minimal system $\{Y_i\}_1^\infty$ with a total conjugate system $\{\Phi_i\}_1^\infty$. From each adjacent class Y_i choose one element $y_i \in E$, and to each linear functional Φ_i assign the linear functional φ_i generated by it, defined in E:

$$\varphi_i(x) = \Phi_i(x + E_1) \quad (x \in E).$$

It is easy to verify that the set $\{x_i\}_1^\infty \cup \{y_i\}_1^\infty$ is complete in E, and the set $\{f_i\}_1^\infty \cup \{\varphi_i\}_1^\infty$ is total. With the aid of the equalities

$$z_n = y_n + \sum_{i=1}^{n} \lambda_{ni} x_i; \qquad \psi_n = f_n + \sum_{i=1}^{n} \mu_{ni} \varphi_i \quad (n = 1,\ 2, ...),$$

where

$$\lambda_{ni} = -f_i(y_n); \quad \mu_{ni} = -f_n(y_i); \quad \lambda_{nn} + \mu_{nn} = 1 - f_n(y_n),$$

we define new systems $\{z_i\}_1^\infty$ and $\{\psi_i\}_1^\infty$. A direct calculation shows that the set $\{x_i\}_1^\infty \cup \{z_i\}_1^\infty$ is a complete minimal system in E with total conjugate system $\{\psi_i\}_1^\infty \cup \{\varphi_i\}_1^\infty$.

Theorem 2. *Let $\{x_i\}_1^\infty \cup \{y_i\}_1^\infty$ be a minimal system in E, and suppose that $\{x_i\}_1^\infty$ is complete in the subspace $P \subset E$. Form a new minimal sequence of elements*

$$u_i = \alpha_i x_i + y_{k_i} \quad (i = 1,\ 2, ...),$$

where α_i are arbitrary numbers, and $\{y_{k_i}\}_{i=1}^\infty$ is some subset of the system $\{y_i\}_1^\infty$. Denote by R the closure of the linear span of the system $\{u_i\}_{i=1}^\infty$. If the system conjugate to $\{u_i\}_1^\infty$ is total with respect to R, then $R \cap P = \theta$.

Proof. Let x be an arbitrary element of the closure of the linear span of the system $\{x_i\}_1^\infty \cup \{y_i\}_1^\infty$:

$$x = \lim_{n\to\infty} \left\{ \sum_{i=1}^n a_{ni} x_i + \sum_{i=1}^n b_{ni} y_i \right\}. \tag{1}$$

Since this system is minimal, the limits exist

$$\lim_{n\to\infty} a_{ni} = a_i(x); \qquad \lim_{n\to\infty} b_{ni} = b_i(x) \quad (i = 1, 2, ...). \tag{2}$$

Suppose now that $x \in P \cup R$. From the fact that $x \in P$, it follows that $b_i(x) = 0$ $(i = 1, 2, ...)$, and from the fact that $x \in R$, it follows that

$$x = \lim_{n\to\infty} \sum_{i=1}^n c_{ni} u_i = \lim_{n\to\infty} \left\{ \sum_{i=1}^n c_{ni} \alpha_i x_i + \sum_{i=1}^n c_{ni} y_{k_i} \right\}. \tag{3}$$

Comparing (1), (2), and (3), we see that

$$\lim_{n\to\infty} c_{ni} = b_i = 0 \quad (i = 1, 2, ...);$$

since the system $\{u_i\}_1^\infty$ has a total conjugate, the last equality implies $x = \theta$.

Remark 1. The system $\{u_i\}_1^\infty$ has a total conjugate, for example, in the case when the system $\{x_i\}_1^\infty \cup \{y_i\}_1^\infty$ has this property.

Remark 2. Theorems 1 and 2 complement the results of V. G. Vinokurov ([2,3]) and generalize some of them.

We shall call Banach spaces E_1 and E_2 ε-isometric if there exists an isomorphism T from E_1 onto E_2 such that for every $x \in E_1$:

$$(1-\varepsilon)\|x\| \leq \|Tx\| \leq (1+\varepsilon)\|x\| \quad (0 < \varepsilon < 1).$$

Theorem 3. Let $\{x_i\}_1^\infty$ be a complete minimal system in a subspace $P \subset E$. For a given $\varepsilon > 0$ there exists a sequence of positive numbers ε_i such that, for any sequence $\{y_i\}_1^\infty$, $y_i \in E$, satisfying the condition

$$\|x_i - y_i\| < \varepsilon_i \quad (i = 1, 2, ...),$$

the linear mapping T taking x_i to y_i $(i = 1, 2, ...)$ will be an ε-isometry of the subspace P onto the closure of the linear span of $\{y_i\}_1^\infty$.

Proof. Put $\varepsilon_i = \varepsilon r_i / 2^i$, where r_i is the distance of the element x_i from the linear span of the remaining elements of the system $\{x_i\}_1^\infty$. Define on the set of linear combinations of the elements of this system a linear operator T, putting

$$Tx_i = y_i \quad (i = 1, 2, ...).$$

From the inequality

$$\|x\| = \left\|\sum_{i=1}^n \alpha_i x_i\right\| \geq |\alpha_i|\, r_i \quad (i = 1, 2, ...)$$

it follows that

$$|\alpha_i| \leq \frac{\|x\|}{r_i}.$$

Let us now estimate $\|Tx\|$:

$$\|Tx\| = \left\|\sum_{i=1}^n \alpha_i y_i + \sum_{i=1}^n \alpha_i (x_i - y_i)\right\|;$$

since

$$\left\|\sum_{i=1}^n \alpha_i (x_i - y_i)\right\| \leq \sum_{i=1}^n \frac{\|x\|}{r_i} \cdot \frac{\varepsilon r_i}{2^i} \leq \varepsilon \|x\|,$$

it follows that

$$(1-\varepsilon)\|x\| \leq \|Tx\| \leq (1+\varepsilon)\|x\|.$$

Extending the operator T by continuity to all of P, we obtain the required ε-isometric mapping.

Let P and Q be subspaces in E, with $P \cap Q = \theta$. We shall call the sum of P and Q a quasi-direct sum and denote by $P \dotplus Q$ the closure of the set of elements of the form $x + y$ $(x \in P,\ y \in Q)$. If $P \dotplus Q = E$, then each of the subspaces P and Q is called a quasicomplement of the other in E.

Theorem 4. If P and Q are nontrivial subspaces in E and $P \supset Q$, then for any $\varepsilon > 0$:

1) There exists a subspace $\widetilde{Q} \subset E$, ε-isometric to Q and quasicomplementary to P.
2) There exists a subspace $\widetilde{P} \subset E$, ε-isometric to P and quasicomplementary to Q.

Proof. By Theorem 1, in E there exists a complete minimal system $\{x_i\}_1^\infty \cup \{y_i\}_1^\infty \cup \{z_i\}_1^\infty$, with conjugate system total relative to E, such that $\{x_i\}_1^\infty$ is a complete system in Q, and $\{x_i\}_1^\infty \cup \{y_i\}_1^\infty$ is complete in P. By Theorem 3 one can choose a sequence $\{\varepsilon_i\}_1^\infty$, $\varepsilon_i > 0$, so that the closure $\widetilde{Q}$ of the linear span of the elements $u_i = x_i + \varepsilon_i z_i$ is ε-isometric to Q. Since the system $\{u_i\}_1^\infty$ has, evidently, a conjugate system total (relative to $\widetilde{Q}$), by Theorem 2 $\widetilde{Q} \cap P = \theta$. Moreover $P \dotplus \widetilde{Q}$ contains all elements of the set $\{x_i\}_1^\infty \cup \{y_i\}_1^\infty \cup \{z_i\}_1^\infty$, and therefore $P \dotplus \widetilde{Q} = E$, which proves assertion 1). Assertion 2) is proved similarly.

The theorem proved generalizes Mackey' s result (4) on the existence of a quasicomplement to any subspace in a separable Banach space.

Corollary 1. If P is a nontrivial subspace in E, then for any $\varepsilon > 0$, $E = P \dotplus \widetilde{P}$, where $\widetilde{P}$ is ε-isometric to P.

Since in every Banach space there exists an infinite-dimensional subspace with a basis, it follows directly from Theorem 4 that

Theorem 5. In a separable Banach space every infinite-dimensional subspace has a quasicomplement with a basis.

Corollary 2. A separable Banach space E can be represented as the quasi-direct sum of its subspaces with bases.

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Note: Figure translations are in progress. See original paper for figures.

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