Mathematics

1962

SovietRxiv

Abstract

Full Text

Mathematics

L. B. Shneperman

Semigroups of Continuous Transformations

(Presented by Academician A. I. Mal'tsev on 8 I 1962)

1°. In this work it is established that a completely regular topological space containing a simple arc is completely characterized by the semigroup of its continuous transformations (7°). An abstract characterization is given of the semigroup of all continuous transformations of a bounded metric space as a topological semigroup (12°)*, and, for the case of a bounded metric space containing a simple arc, also as a semigroup (16°). It is shown that a topology can be introduced into this latter semigroup in an intrinsic way so that it coincides with the natural one (15°). Analogous problems are also solved for a bounded closed set on the real line (19°, 21°, 22°).

2°. The necessary facts from the general theory of semigroups are contained in ([1]), and from the theory of metric spaces in ([2]).

3°. A semigroup $\mathfrak{A}$, on whose set of elements a topology t is defined, will be denoted by $\mathfrak{A}_t$. In the case when the semigroup operation is continuous with respect to this topology, the semigroup $\mathfrak{A}_t$ is called topological. If the topology on the set of elements of the semigroup is generated by a metric r, then we shall denote this semigroup by $\mathfrak{A}_r$, and, in the case of continuity of the semigroup operation with respect to this metric, the topological semigroup $\mathfrak{A}_r$ is called metric. A subsemigroup, supersemigroup, topological subsemigroup, and topological supersemigroup of the semigroup $\mathfrak{A}_t$ are defined in the natural way. An isomorphism of semigroups with topology is understood as an isomorphism of structures in the sense of ([3]).

4°. Let Ω be the minimal two-sided ideal of the semigroup $\mathfrak{A}$. Define a family Γ of subsets of the set of elements of Ω. $\mathfrak{A} \in \Gamma$ if and only if there exist elements $L \in \Omega$ and $A \in \mathfrak{A}$ such that $A\mathfrak{A} = L$ and $A\Omega \setminus \mathfrak{A} \supset L$.

We take the family Γ as a pseudobase of the closed sets of a topology on Ω ([4]). Everywhere below this topology will be denoted by τ.

It is clear that in this way one can construct a topology on any left ideal of the semigroup $\mathfrak{A}$, but here this construction is used only for constructing a topology on the minimal two-sided ideal.

5°. Let Ω be a topological space and $\mathfrak{C} = \mathfrak{C}(\Omega)$ the set of all its continuous transformations. The set $\mathfrak{C}$ is a semigroup with respect to the operation of superposition of transformations. By $\mathfrak{H} = \mathfrak{H}(\Omega)$ we shall denote the set of all transformations of Ω mapping all points of Ω into one and the same point. It is clear that $\mathfrak{H}$ is a subsemigroup and is contained in $\mathfrak{C}$. It is easy to show that $\mathfrak{H}$ is the minimal two-sided ideal of the semigroup $\mathfrak{C}$.

To each point $\alpha \in \Omega$ we associate the transformation $H_\alpha \in \mathfrak{H}$ mapping the space Ω into this point. This one-to-one mapping of Ω onto $\mathfrak{H}$ induces on the set of elements of $\mathfrak{H}$ a topology, which will be denoted by t_0. The homeomorphism of the topological spaces $\mathfrak{H}_{t_0}$ and Ω is obvious.

---

* This result, as well as the assertions of item 8°, is close to results of L. M. Gluskin ([6]).

6°. In this and in the following item, Ω, as well as Ω', are completely regular topological spaces (5°) containing spaces (5°) that contain a simple arc.

Lemma. *On the minimal two-sided ideal $\mathfrak{H}$ of the semigroup $\mathfrak{C}$, the topology t_0 (5°) coincides with τ (4°).*

7°. **Theorem.** *In order that the semigroups $\mathfrak{C}(\Omega)$ and $\mathfrak{C}(\Omega')$ be isomorphic, it is necessary and sufficient that the topological spaces Ω and Ω' be homeomorphic.*

8°. We shall call the left ideal Ω_r of the semigroup $\mathfrak{A}_r$ (3°) an r-ideal if, for all $A, B \in \mathfrak{A}$,

$$r(A, B) = \sup_{L \in \Omega} r(AL, BL).$$

Denote by $M(\Omega_r)$ the class of all metric semigroups containing Ω_r as an r-ideal. If the class of semigroups $M(\Omega_r)$ is nonempty and Ω_r is compact, then one can prove that in this class, ordered by inclusion, there is a unique (up to isomorphism) maximal element. We shall call it the r-oversemigroup of the metric semigroup Ω_r. It is not difficult to show that if a topological semigroup is compact and metrizable so that it is its own r-ideal, then under any metrization the r-oversemigroups of the corresponding metric semigroups are isomorphic. It is therefore natural to speak of the r-oversemigroup of such a topological semigroup.

9°. We shall say that a semigroup $\mathfrak{A}$ belongs to the class of semigroups Σ if, for any $A, B \in \mathfrak{A}$, $AB = A$.

The semigroup $\mathfrak{H}$ (5°) belongs to the class of semigroups Σ.

10°. Let Ω be a bounded metric space. On the semigroup $\mathfrak{C}$ (15°) define the metric r_0, starting from the metric r on Ω: for each pair of transformations $S_1, S_2 \in \mathfrak{C}$,

$$r_0(S_1, S_2) = \sup_{\xi \in \Omega} r(S_1\xi, S_2\xi).$$

This metric induces on $\mathfrak{C}$ the topology t_0. The notation $\mathfrak{H}_{t_0}$ and $\mathfrak{C}_{t_0}$ is explained in 5°.

The topological spaces $\mathfrak{H}_{t_0}$ and Ω are homeomorphic. This explains the fact that for the topology the same notation is chosen as in 5°.

11°. **Lemma.** *Let Ω be compact. Then the semigroup $\mathfrak{C}_{t_0}$ (10°) is topological and is the r-oversemigroup of the topological semigroup $\mathfrak{H}_{t_0}$ (8°).*

12°. The assertions stated in 9°–11° make it possible to give an abstract characterization of the topological semigroup $\mathfrak{C}_{t_0}$ (10°).

Theorem. *Let Ω be compact. A topological semigroup $\mathfrak{B}_t$ is isomorphic to the topological semigroup $\mathfrak{C}_{t_0}(\Omega)$ if and only if:*

1) *$\mathfrak{B}$ contains a minimal two-sided ideal $\mathfrak{K}$;*

2) *the semigroup $\mathfrak{K}$ belongs to the class of semigroups Σ (9°);*

3) *the topological space $\mathfrak{K}_t$ is homeomorphic to Ω;*

4) *$\mathfrak{B}_t$ is the r-oversemigroup of the topological semigroup $\mathfrak{K}_t$.*

13°. Let us note that if Ω is a bounded metric space containing a simple arc, or a bounded subset of the line, then:

Theorem. *In order that the semigroup $\mathfrak{C}_{t_0}$ (10°) be topological, it is necessary and sufficient that Ω be compact.*

14°. Let Ω be the minimal two-sided ideal of the semigroup $\mathfrak{A}$, and let the topological space Ω_τ (4°) be compact and metrizable so that it is its own r-ideal (8°). If r is such a metric on Ω and there are no elements in the semigroup $\mathfrak{A}$ that act identically on Ω from the left (i.e. the equality $AL = BL$ holds for all $L \in \Omega$ only when $A = B$), then the metric r can be extended to $\mathfrak{A}$: for any $A, B \in \mathfrak{A}$,

$$r(A, B) = \sup_{L \in \Omega} r(AL, BL).$$

One can prove that this metric induces on $\mathfrak{A}$ a topology which does not depend on the manner of metrizing Ω_τ. Therefore it is also denoted by τ. Let us also note that the semigroup $\mathfrak{A}_\tau$ is topological.

15°. **Theorem.** *Let Ω be compact and contain a simple arc. On the semigroup $\mathfrak{C}$ (5°), the topologies τ (14°) and t_0 (10°) coincide.*

16°. If Ω satisfies the conditions of Theorem 15°, then from the abstract characteristic of the topological semigroup $\mathfrak{C}_{t_0}$ (10°) one can always obtain an abstract characteristic of the semigroup $\mathfrak{C}$ (5°). For this it is enough to introduce on it the topology τ (14°), and then to verify whether the topological semigroup thus obtained satisfies the conditions of the corresponding theorem, for example, Theorem 12°.

17°. Let Ω be a minimal two-sided ideal of the semigroup $\mathfrak{A}$. Let $\mathfrak{Z}$ be an arbitrary subset of the set of its elements, and let L_1, L_2 be an arbitrary but fixed pair of elements of Ω. By $\rho^{\mathfrak{Z}}_{L_1,L_2}$ we denote the binary relation on Ω consisting of all pairs of the form (ZL_1, ZL_2):

$$\rho^{\mathfrak{Z}}_{L_1,L_2} = \bigcup_{Z\in\mathfrak{Z}} (ZL_1, ZL_2).$$

Let now $\mathfrak{R}$ be an idempotent subsemigroup of the semigroup $\mathfrak{A}$, and suppose there exists such a pair of elements $L^*, L^{**} \in \Omega$ that the binary relation $\rho^{\mathfrak{R}}_{L^*,L^{**}}$ is a relation of linear order. It can be shown that the existence of more than one such pair L^*, L^{**} is impossible. Therefore we shall say that the semigroup $\mathfrak{R}$ induces on Ω a relation of linear order, and denote this relation by $\rho^{\mathfrak{R}}$. This relation corresponds to a topology on the set of elements Ω, namely, the topology of the chain (4).

If there exists at least one idempotent subsemigroup $\mathfrak{R}$ of the semigroup $\mathfrak{A}$ inducing on Ω a relation of linear order $\rho^{\mathfrak{R}}$, and if to each such subsemigroup there corresponds one and the same topology on Ω, then we shall say that the minimal two-sided ideal Ω of the semigroup $\mathfrak{A}$ admits the topology of a chain, and denote this topology by λ. This construction, just as in 4°, can be carried out on any left ideal, but we shall not need this.

18°. Here and in all subsequent items, Ω, as well as Ω', are bounded closed sets on the line.

Lemma. *The minimal two-sided ideal $\mathfrak{H}$ of the semigroup $\mathfrak{C}$ (5°) admits the topology of a chain (17°), and on $\mathfrak{H}$ the topologies λ and t_0 coincide.*

19°. **Theorem.** *In order that the semigroups $\mathfrak{C}(\Omega)$ and $\mathfrak{C}(\Omega')$ be isomorphic, it is necessary and sufficient that Ω and Ω' be homeomorphic.*

20°. Let the minimal two-sided ideal Ω of the semigroup $\mathfrak{A}$ admit the topology of a chain (17°), and let the topological space Ω_λ be compact and metrizable in such a way that it is its r-ideal. If every inner left shift of the semigroup $\mathfrak{A}^*$ is a continuous transformation of the topological space Ω_λ, and if in the semigroup $\mathfrak{A}$ there are no elements acting identically on Ω from the left, then the topology λ can be extended to $\mathfrak{A}$ in exactly the same way as in 14°.

21°. **Theorem.** *On the semigroup $\mathfrak{C}$ (5°) the topologies λ (20°) and t_0 (10°) coincide.*

22°. Based on Theorem 21°, in the same way as in 16°, we obtain an abstract characteristic of the semigroup $\mathfrak{C}$.

I take this opportunity to express my gratitude to Prof. E. S. Lyapin for valuable advice and constant attention to the work.

Leningrad State Pedagogical Institute
named after A. I. Herzen

Received
28 XII 1961

REFERENCES

1. E. S. Lyapin, *Semigroups*, Moscow, 1960.
2. F. Hausdorff, *Set Theory*, 1937.
3. N. Bourbaki, *General Topology*, Moscow, 1958.
4. G. Birkhoff, *Lattice Theory*, IL, 1952.
5. L. S. Pontryagin, *Continuous Groups*, Moscow, 1954.
6. L. M. Gluskin, *Mat. sbornik*, **55**, No. 4 (1961).

* Let A be an element of the semigroup $\mathfrak{A}$. The inner left shift of the semigroup $\mathfrak{A}$ is the following transformation φ_A of the set of its elements: for any $x \in \mathfrak{A}$, $\varphi_A x = Ax$.

Note: Figure translations are in progress. See original paper for figures.