ON THE CLASSIFICATION OF FORMAL ABELIAN GROUPS

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Abstract

Full Text

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MATHEMATICS

1. In content, this communication adjoins the note ([1]), whose terminology and notation we shall use below without special reservations. The aim of the communication is to present results on the classification, up to isomorphism, of formal Abelian groups over an algebraically closed field of constants k of characteristic $p > 0$. These results constitute a generalization of the conclusions of note ([1]) to the case of groups of arbitrary dimension in the following restricted sense: we consider only groups whose unipotent component consists solely of additive groups, and about the "space of modules" we obtain somewhat less precise information than in the two-dimensional case.

More specifically, the object of study is the class K of such groups G for which the homomorphism pI of multiplication by p is an isogeny of the group $(pI)G$. Every such group is isogenous to a direct sum of the form $\sum G_{n_i,m_i}$, where either $m_i < \infty$ and $(n_i, m_i) = 1$, or $m_i = \infty$, and then $n_i = 1$. It often turns out to be possible to assume that the multiplicative groups $G_{1,0}$ do not enter into this direct sum, since in any isogenous group they split off as a direct summand and are immaterial for the classification problem. The principal qualitative result of the work is the following theorem: *the classes of isomorphic groups isogenous to a fixed group G of the class K correspond one-to-one to the points of the quotient space of a certain quasi-projective variety (over the field k) by the group of its regular automorphisms.* A more precise formulation is given below; still more precise information about the nature of this "space of modules" and about the method of its effective computation would be difficult to formulate without entering into the details of the proof.

Let us note that, according to our definition, the coordinate ring of a formal group contains no nilpotent elements. Nevertheless, with the aid of Lazard' s results on the prolongation of germs of formal Abelian groups, or directly from considerations of Dieudonné module theory, it follows that our conclusions, with the corresponding modifications, are valid also for a more general class of formal groups with nilpotent elements (see ([2])).

As in note ([1]), all results are formulated in the language of formal groups and their coordinate rings, although in the proofs one has to deal exclusively with the Dieudonné modules of these groups. This corresponds to the author' s conviction that the real content of the theory of formal groups consists in the study of group schemes of a special kind, and not of modules; the latter are only a very convenient auxiliary tool.

2. Theorem 1. *Let G be a group of the class K. There exists a number H, depending only on the group G, such that for any two groups G', G'', isogenous to G, one can find either an isogeny $G' \to G''$, or an isogeny $G'' \to G'$, whose height does not exceed H.*

Remark 1. One can prove that, conversely, the existence of a number H with the indicated property entails that the group belongs to···

G to the class K. In this lies a certain intrinsic justification for the introduction of the class K.

Remark 2. The number H can be written explicitly in terms of the invariants (n_i, m_i) of the decomposition $\sum G_{n_i m_i}$ of the group G up to isogeny.

Theorem 1 makes it possible, in essence, to reduce the problem of classifying groups of the class K to the problem of studying isogenies $G' \to \sum G_{n_i m_i}$ of **bounded height**. The main inconvenience in carrying out this program directly is that a fixed group G' may have several essentially different isogenies $G' \to \sum G_{n_i m_i}$. This difficulty is overcome by introducing the notion of a **special group**.

Definition. A group G, isogenous to the group $kG_{n,m}$, $m \neq \infty$, is called special if $(p^{kn} I)\mathfrak{o} = \mathfrak{o}^{p^{k(m+n)}}$. A group G not containing a unipotent component (i.e. equidimensional in Barsotti’ s terminology) is called special if there exists a finite number of homomorphisms $\varphi_i : G \to G_i$, where the groups G_i are isogenous to $k_i G_{n_i m_i}$ and are special, and the images $\varphi_i(\mathfrak{o}_i) \subset \mathfrak{o}$ of the coordinate rings of the groups G_i generate the entire coordinate ring $\mathfrak{o}$ of the group G. Finally, a group of the class K is called special if it is isomorphic to the direct sum of a special equidimensional group and a finite number of additive groups.

Closely connected with Theorem 1 is the first important property of special groups:

Theorem 2. *For any group G of the class K there exists only a finite number of special groups isogenous to G and pairwise nonisomorphic.*

(In fact, one obtains a fairly explicit description of the special groups and an algorithm for listing them.)

The significance of special groups for the classification of groups of the class K is clarified by the following circumstance:

Theorem 3. *For any group G of the class K, the class of all its isogenies $\varphi : G \to G_0$ onto special groups G_0 is nonempty, and in it there exists a unique minimal isogeny (with respect to the order relation: $\varphi'' > \varphi'$ if $\varphi'' = \varepsilon \circ \varphi'$, $\varepsilon : G_0 \to G_0'$; uniqueness is understood in the natural sense).*

Now the classification of groups of the class K is carried out in several steps. The first step consists in decomposing the class K into classes of isogenous groups. Each such class contains a group of the form $\sum G_{n_i m_i}$ and is determined uniquely, up to permutation of the pairs (n_i, m_i), by the set of these pairs.

At the second step, the class of isogenous groups is divided into a finite number of subclasses: the groups G' and G'' belong to the same subclass if the special groups G'_0 and G''_0, defined by the minimal isogenies $G' \to G'_0$, $G'' \to G''_0$, are isomorphic. (For groups isogenous to $G_{2,m}$, the subclass corresponds to a fixed value of the difference $h - j$; see (1), Lemma 3.)

Fix a special group G_0. The third and final step of the classification—the description of the groups in the subclass determined by G_0—can be summarized in the following statement:

Theorem 4. A. *The set of equivalence classes of minimal isogenies $G \to G_0$ can be endowed with the structure of a quasiprojective variety $A = A(G_0)$ over the field k. (This variety is, generally speaking, reducible, mixed, and incomplete.)*

B. *The automorphism group Γ of the formal group G_0 acts on the variety A as the group of its regular automorphisms, and the stable normal divisor has finite index if the group G_0 is equidimensional.*

The set of isomorphism classes of formal groups minimally isogenous to G_0 is in one-to-one correspondence with the set of orbits A/Γ.

3. Apart from the incompleteness of the information about the spaces A, which one may hope to overcome by improving computational techniques, the result formulated has the following shortcoming: we refrain from describing the natural topology on the union $\bigcup_{G_0} A(G_0)$ and, still more, on the "space of moduli" $\bigcup A(G_0)/\Gamma(G_0)$. As I. R. Shafarevich pointed out to the author, this question is connected with the absence of a general definition of what is meant by the "quotient space of an algebraic set by an algebraic group acting on it" in the case when the orbits are not closed and have different dimensions.

4. The question of the "space of moduli" for groups isogenous to a group that contains a unipotent component with Witt groups remains open. The example $G_{2,\infty}$ shows that this space consists of an infinite set of components, for the distinction of which, in the general case, one must find some new infinite-valued invariants.

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REFERENCES

[1] Yu. I. Manin, DAN, **143**, No. 1 (1962). [2] P. Gabriel, Séminaire Serre, 1960.

Note: Figure translations are in progress. See original paper for figures.