V. A. Gaukman

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Abstract

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V. A. Gaukman

Symmetric Almost Symplectic Affine Connections on Hermitian Manifolds

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Let $\mathfrak{M}$ be a $2n$-dimensional Hermitian manifold (^{1}), i.e., on $\mathfrak{M}$ there are given a tensor F^i_j and a positive definite symmetric tensor g_{ij}, satisfying the conditions:

$$F^i_k F^k_j = -\delta^i_j; \quad g_{lm} F^l_i F^m_j = g_{ij}; \quad t^i_{jk} = \frac{1}{4}\left(F^l_j \frac{\partial F^i_k}{\partial x^l} - F^l_j \frac{\partial F^i_l}{\partial x^k} - F^l_k \frac{\partial F^i_j}{\partial x^l} + F^l_k \frac{\partial F^i_l}{\partial x^j}\right) = 0.$$

The tensor $a_{ij} = -F^l_i g_{lj}$ is skew-symmetric and

$$a_{lm} F^l_i F^m_j = a_{ij}. \tag{1}$$

The quantities

$$T_{ijk} = \frac{1}{3}\left(\partial a_{ij}/\partial x^k + \partial a_{jk}/\partial x^i + \partial a_{ki}/\partial x^j\right) = \frac{1}{3} a_{(ijk)},$$

where parentheses denote cyclic summation and $a_{ijk} = \partial a_{ij}/\partial x^k$, define a tensor whose vanishing means that the space is Kähler.

In the present note we shall find a **symmetric** affine connection invariantly associated with the tensors F^i_j, g_{ij}, satisfying the conditions indicated above.

We note that on Hermitian manifolds various canonical affine connections have already been constructed (see, for example, (^{1,2})); however, all of them except the Frölicher connection (^{2}) are not symmetric. The Frölicher connection is almost complex, i.e., $F^i_{j,k} = 0$ (the comma here and below denotes covariant differentiation). We set ourselves the goal of finding a symmetric affine connection that is not almost complex, but instead belongs to the class of so-called almost symplectic connections (^{3}).

It is not difficult to show that the condition $a_{ij,k} = 0$ implies $T_{ij,k} = 0$. However, V. G. Lemlein (^{3}) showed that there exists a pencil of symmetric affine connections satisfying the condition $a_{ij,k} = T_{ijk}$. The components of these connections have the form:

$$\Gamma^i_{jk} = \frac{1}{3} a^{il}\left(a_{ljk} - a_{klj} + \gamma_{ljk}\right), \tag{2}$$

where $a^{il} a_{lj} = \delta^i_j$, and γ_{ijk} is an arbitrary geometric object symmetric in each pair of indices, whose transformation law for its components under a change of local coordinates can be obtained from the equality $\gamma_{ijk} = -a_{l(i}\Gamma^l_{jk)}$. Lemlein called these connections almost symplectic.

Let us require that the desired connection be almost symplectic. Our problem is now reduced to the choice of the components of the object γ_{ijk}.

To facilitate the further calculations it is useful to note that on a Hermitian manifold there exist so-called adapted systems of coordinates, in which the components of the tensor F^i_j are constant (^{1}). Passing to an adapted coordinate system, differentiation of equality (1) now gives

$$F^l_i F^m_j a_{lmk} = a_{ijk}. \tag{3}$$

Consider the tensor $F^i_{j,k}$. It has two pure (1)-components (see (4)):

$$A^i_{jk} = -(F^l_j F^i_{l,k} + F^l_k F^i_{j,l}), \qquad B^i_{jk} = F^l_j (F^i_{k,l} - F^i_{l,k}).$$

It can be verified that the identity

$$T_{ijk} = \frac{1}{2} F^l_{(i} F^n_j B^m_{k)m} a_{tl}$$

holds, from which it follows that the vanishing of the tensor B^i_{jk} entails the vanishing of the tensor T_{ijk}.

We now require that, for the desired connection, the equality

$$A^i_{jk} = 0. \tag{4}$$

hold. This gives:

$$\gamma_{ijk} - F^l_{(i} F^m_j \gamma_{k)lm} = 0. \tag{5}$$

For what follows we shall need the notion of a conjugate affine connection (4). Let Γ^i_{jk} be a symmetric affine connection. By the conjugate affine connection $\tilde{\Gamma}^i_{jk}$ we shall mean the connection in which parallel transport from a point x_1 to a point x_2 along a curve L is carried out in the following way: the vector is first subjected to the automorphism $F(x_1) = (F^i_j(x_1))$, then transported from x_1 to x_2 along L parallel with respect to the connection Γ^i_{jk}, and finally subjected to the inverse automorphism $F^{-1}(x_2) = -F(x_2)$. The components of this connection have the form $\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + F^l_j F^i_{l,k}$.

We now require that the equality

$$g_{ij,k} = g_{ij|k}, \tag{6}$$

hold, where $g_{ij|k}$ is the covariant derivative of the tensor g_{ij} in the connection $\tilde{\Gamma}^i_{jk}$. We obtain

$$\gamma_{ijk} - F^l_i F^m_j \gamma_{klm} = \frac{1}{2}(a_{kji} + a_{kij} - F^l_i F^m_j a_{klm} - F^l_i F^m_j a_{kml}),$$

whence it follows:

$$3\gamma_{ijk} - F^l_{(i} F^m_j \gamma_{k)lm} = -\frac{1}{2} F^l_{(i} F^m_j a_{k)lm} - \frac{1}{2} F^l_{(i} F^m_j a_{k)ml}. \tag{7}$$

From (5) and (7) we have

$$\gamma_{ijk} = -\frac{1}{2} F^l_{(i} F^m_j a_{k)lm}. \tag{8}$$

We must now verify that the connection (8) indeed satisfies conditions (4) and (6). First note that the identity

$$T_{ijk} = F^l_{(i} F^m_j T_{k)lm}, \tag{9}$$

holds; for its proof it is sufficient to use equality (3). By direct computation we obtain

$$6a_{il} F^l_{j,k} = F^l_j a_{ikl} - F^l_i a_{jkl} + F^l_i a_{lkj} - F^l_j a_{lki}.$$

Hence

$$a_{il} F^l_{j,k} + a_{jl} F^l_{i,k} = 0. \tag{10}$$

Further:

$$g_{ij,k} = F^l_{i,k} a_{lj} + F^l_i a_{lj,k}. \tag{11}$$

Interchanging i and j and adding, we obtain

$$g_{ij,k} = \frac{1}{2}(F^l_i T_{jkl} + F^l_j T_{ikl}). \tag{12}$$

From (11) we have

$$F^i_{j,k} = a^{li} g_{jl,k} - F^i_j a^{mt} T_{lmk},$$

whence

$$F^i_{j,k} = \frac{1}{2} a^{li} (F^m_l T_{jkm} - F^m_j T_{lkm}).$$

Now a computation gives

$$a_{il} A^l_{jk} = \frac{1}{2} T_{ijk} - \frac{1}{2} F^l_{(i} F^m_j T_{k)lm},$$

which, by virtue of (9), proves (4). To prove (6), note that

$$g_{ij,k} - g_{ij|k} = -(a_{il} F^l_{j,k} + a_{jl} F^l_{i,k}) = 0$$

by virtue of (10). Thus, it has been proved that the connection (8) satisfies conditions (4) and (6).

Using the method of prolongation and envelopments of geometric objects of G. F. Laptev $^{(5)}$, extended by A. M. Vasil' ev $^{(6)}$ to the case of infinite transformation groups, one can show that the expressions (2), in which

$$\gamma_{ijk} = -\frac{1}{2} F^l_{(i} F^m_j a_{k)lm} - \frac{1}{2} F^m_{(i} a_{j|l} F^l_{k)m} + \frac{1}{2} F^m_{(j} a_{i|l} F^l_{mk)} + \frac{1}{2} F^m_{(i} a_{j|l} F^l_{mk)}, \qquad (13)$$

where $F^i_{jk} = \partial F^i_j / \partial x^k$, and there is no cyclic summation over the underlined indices, define the components of an object of affine connection. For an adapted coordinate system $F^i_{jk} = 0$, and therefore expression (13) coincides with (8). Thus formula (13) makes it possible to compute the components of our connection for an arbitrary coordinate system. Hence the following has been proved.

Theorem 1. *On a Hermitian manifold there exists, and moreover uniquely, a symmetric affine connection satisfying the conditions:*
1) $a_{ij,k} = T_{ijk}$*; 2)* $A^i_{jk} = 0$*; 3)* $g_{ij,k} = g_{ij|k}$*. The components of this connection are determined by formulas (2) and (13).*

In $^{(4)}$ the so-called normal connections were considered, i.e. connections for which $B_i = B^l_{il} = 0$. Such connections play a role in the study of families of geodesic lines. For our connection $B_i = -\frac{1}{2} a^{lm} T_{ilm}$. The vanishing of this vector means $^{(3)}$ that the volume

$$\sqrt{\det \|a_{ij}\| \, \det \|X^k_{(l)}\|}$$

is invariant. But

$$\det \|a_{ij}\| = \det \|g_{il} F^l_j\| = \det \|g_{ij}\| \times \det \|F^i_j\| = \pm \det \|g_{ij}\|.$$

Consequently, we have proved:

Theorem 2. *In order that the almost symplectic connection (13) be normal, it is necessary and sufficient that the space have the invariant volume* $\sqrt{\det\|g_{ij}\| \,\det\|X^k_{(l)}\|}$.

Denote $B_{ijkl} = a_{im}R^m_{jkl}$, where

$$R^i_{jkl} = \partial\Gamma^i_{jl}/\partial x^k - \partial\Gamma^i_{jk}/\partial x^l + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml}$$

is the curvature tensor of the connection (13). From the general properties of the curvature tensor there follow the relations

$$B_{ijkl} = -B_{ijlk}; \tag{I}$$

$$B_{i(jkl)} = 0. \tag{II}$$

For almost symplectic connections one additionally has [(3)]

$$B_{ijkl} + B_{jilk} + B_{klij} + B_{lkji} = 0. \tag{III}$$

For our connection (13) we have, in addition,

$$\begin{gathered} B_{ijkl} - F^r_j F^s_k B_{irsl} - F^r_j F^s_l B_{irks} - F^r_k F^s_l B_{ijrs} - F^r_i F^s_j B_{rskl} \\ - F^r_i F^s_k B_{rjsl} - F^r_i F^s_l B_{rjks} + F^r_i F^s_j F^p_k F^q_l B_{rspq} = 0; \end{gathered} \tag{IV}$$

$$\begin{gathered} B_{(i|jk)} - F^r_i F^s_j B_{(k|rs)} - F^r_k F^s_i B_{(j|rs)} - F^r_j F^s_k B_{(i|rs)} \\ = -2F^r_{(i,l}F^s_j T_{k)rs} - 2F^r_{(i}F^s_{j,l}T_{k)rs}; \end{gathered} \tag{V}$$

$$\begin{aligned} &4F^m_j B_{mikl} + 4F^m_i B_{mjlk} + F^m_i B_{(jklm)} + F^m_j B_{(iklm)} - F^m_i B_{(jlkm)} - F^m_j B_{(ilkm)} \\ &\qquad = 2F^m_{i,l}T_{jkm} + 2F^m_{j,l}T_{ikm} - 2F^m_{i,k}T_{jlm} - 2F^m_{j,k}T_{ilm}. \end{aligned} \tag{VI}$$

(IV) follows from the relation $A^i_{jk} = 0$. (V) is obtained by covariant differentiation of the identity $T_{ijk} = F^l_{(i}F^m_j T_{k)lm}$ with the use of the relation (see [(3)]) $T_{ijk,l} = -\frac{1}{2}B_{(i|jk)l}$. To obtain (VI), one must, with the aid of (12), compute $\nabla_k\nabla_l g_{ij} - \nabla_l\nabla_k g_{ij}$ and use the Ricci identity.

If one symmetrizes the connections of the works (^{1,2}), then one obtains symmetric affine connections defined on a Hermitian manifold. The connection (2), (13) can coincide with one of these connections only in the Kähler case. However, there is a deep connection between the connection constructed by us and the connections of the works (^{1,2}). Let us introduce the notion of an associated almost symplectic connection: let Γ^i_{jk} be an arbitrary affine connection. Then the quantities $\gamma_{ijk} = -\frac{1}{2}a_{l(i}\Gamma^l_{jk)} - \frac{1}{2}a_{l(i}\Gamma^l_{kj)}$ define an object symmetric in any pair of indices, whose substitution into (2) determines some symmetric almost symplectic affine connection. It is called the connection associated with the connection Γ^i_{jk}. It turns out that the following holds:

Theorem 3. The connection (2), (13) is an almost symplectic affine connection associated with the Riemannian connection $\left\{ {i \atop jk} \right\}$ corresponding to the tensor g_{ij}, as well as with any of the connections of the works (^{1,2}).

Let us also note that our connection is expressed in the following way through the components of the Riemannian connection $\left\{ {i \atop jk} \right\}$:

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + \frac{1}{3}a^{il}a_{lj/k} + \frac{1}{3}a^{il}a_{lk/j},$$

where $a_{ij/k}$ is the covariant derivative of the tensor a_{ij} in the Riemannian connection $\left\{ {i \atop jk} \right\}$.

In conclusion, let us note that if the manifold $\mathfrak{M}$ is regarded as almost Hermitian, i.e., if one does not assume that the tensor

$$t^i_{jk} = \frac{1}{4}\left(F^l_j \partial F^i_k/\partial x^l - F^l_j \partial F^i_l/\partial x^k - F^l_k \partial F^i_j/\partial x^l + F^l_k \partial F^i_l/\partial x^j\right),$$

is zero, then the quantities (2), (13) still define an affine connection; however, relations (4) and (6) now take the form:

$$A^i_{jk} = 2t^i_{jk} + \frac{2}{3}a^{im}a_{jl}t^l_{mk} + \frac{2}{3}a^{lm}a_{kl}t^l_{mj};$$

$$g_{ij,k} - g_{ij/k} = -\frac{4}{3}\left(g_{il}t^l_{jk} + g_{jl}t^l_{ik}\right).$$

The vector $B_i = B^l_{il}$ has the same form as before: $B_i = -\frac{1}{2}a^{lm}T_{ilm}$, and therefore Theorem 2 holds also for almost Hermitian manifolds.

Moscow State University
named after M. V. Lomonosov

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