Reports of the Academy of Sciences of the USSR

1962

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR

1962. Vol. 142, No. 4

MATHEMATICS

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CAUCHY-TYPE INTEGRALS AND BOUNDARY-VALUE PROBLEMS FOR GENERALIZED ANALYTIC FUNCTIONS ON CLOSED RIEMANN SURFACES

(Presented by Academician I. N. Vekua on 22 IX 1961)

On a closed Riemann surface R of genus $p > 2$, consider the differential equation

$$\bar{U}_z = B(z)\overline{U}. \tag{1}$$

Here $B(z)$ is a covariant with respect to $\bar{z}$, continuous on R everywhere except for at most a finite number of points and lines of discontinuity.

In a finite (2) domain T, equation (1) is reduced to the integral equation (1,5)

$$U(z) + \frac{1}{\pi}\iint_T B(t)\overline{U(t)}A(t,z)\,dT = \Phi(z), \tag{2}$$

where $\Phi(z)$ is a function analytic in T. $A(t,z)$ is the Cauchy kernel described in (7), with its singularities removed to $R - T$ *. In contrast to the planar case, equation (2) generally has eigenfunctions (7). However, for any coefficient one can choose a kernel $A(t,z)$ such that equation (2) will have no eigenfunctions **. In this case the number of poles with respect to z (the set δ of these points we call the characteristic divisor) turns out, generally speaking, to be greater than p. Choose arbitrarily in $R - T$ $p-1$ points $P[t_i]$ $(i = 1, 2, \dots, p-1)$ and construct Abelian covariants of the first kind $Z_j'(t)$ $(j = 1, 2, \dots, p-1)$ such that

$$Z_j'(t_i) = \delta_i \quad (i, j = 1, 2, \dots, p-1).$$

Introduce the new kernel

$$\widetilde{A}(t,z) = A(t,z) - \sum_{j=1}^{p-1} Z_j'(t) A(t_i, z), \tag{3}$$

which has the property that $\widetilde{A}(t_i, z) = 0$ $(i = 1, 2, \dots, p-1)$. The covariants $Z_j'(t)$ can moreover be chosen so that the equation

$$U(z) + \frac{1}{\pi} \iint_T B(t)\overline{U(t)}\widetilde{A}(t,z)\, dT = 0 \tag{4}$$

has no eigenfunctions. In what follows we consider only the kernel (3), and denote it simply by $A(t,z)$. We call the equation

$$V_z + \overline{B(z)}\,\overline{V} = 0 \tag{5}$$

conjugate to (1).

* The properties of the kernel $A(t,z)$ have been studied in greater detail in joint work of S. Ya. Gusman and the author, in which, in particular, a kernel with a characteristic divisor of arbitrary order, $\operatorname{ord}(\delta) \geq p$, was constructed. In this case the necessary and sufficient condition $\dim(W - \delta) = 0$ must be satisfied (unpublished).

** Work by S. Ya. Gusman is devoted to this question (unpublished).

for covariants. Solutions of equation (1) (or (5)) that are regular up to poles are called belonging to equation (1) (respectively (5)). Functions belonging to equation (1) and regular everywhere on R we call generalized constants. We denote their number by k_0. As will be shown in our next paper, $0 \leq k_0 \leq 2$. Covariants belonging to equation (5) and regular everywhere on R, we shall call generalized covariants of the first kind. We denote their number by k_1.

Following I. N. Vekua (^{1}), in the domain T Cauchy kernels are constructed satisfying the systems

$$\left.\begin{aligned} \Omega_1(t,z) + \frac{1}{\pi}\iint_T B(\tau)\,\overline{\Omega_2(t,\tau)}\,A(\tau,z)\,dT &= A(t,z),\\ \Omega_2(t,z) + \frac{1}{\pi}\iint_T B(\tau)\,\overline{\Omega_1(t,\tau)}\,A(\tau,z)\,dT &= 0; \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} \Omega_1(t,z) + \frac{1}{\pi}\iint_T \overline{B(\tau)}\,\Omega_2(\tau,z)\,A(t,\tau)\,dT &= A(t,z),\\ \Omega_2(t,z) + \frac{1}{\pi}\iint_T B(\tau)\,\Omega_1(\tau,z)\,\overline{A(t,\tau)}\,dT &= 0. \end{aligned}\right\}$$

There are integral Cauchy formulas for equations (1) and (5). The corresponding integrals of Cauchy type have the form

$$U(z) = \frac{1}{2\pi i}\int_\Gamma \varphi(t)\Omega_1(t,z)\,dt - \overline{\varphi(t)}\,\Omega_1(\tau,z)\,d\bar{t}; \tag{7}$$

$$V(z) = -\frac{1}{2\pi i}\int_\Gamma \omega(\tau)\Omega_1(z,\tau)\,d\tau - \overline{\omega(\tau)}\,\overline{\Omega_2(z,\tau)}\,d\tau, \tag{8}$$

where in the second integral the density is taken to be covariant.

Under the usual assumptions concerning the contour and the densities, the formulas of Yu. V. Sokhotskii hold.

With the aid of equations (6), the latter are analytically continued to the domain $R - T$. Therefore the integral (7) is a function analytic in T^-, with characteristic divisor δ of the kernel $A(t,z)$. The integral (8) is a covariant analytic in $R - T$ with a pole at the point P_0 (see (8)) and zeros at the selected points P_i ($i = 1, \dots, p-1$).

Let the contour Γ bound on R a connected domain T^+, and let T^- be its complement on R.* In the domains $T^\pm$ we construct kernels $\Omega_{1,2}(t,z)$, whose singularities we place in $T^\pm$.

Consider the boundary-value problem: to determine solutions $U^\pm$, regular in $T^\pm$, of equation (1), satisfying the boundary condition

$$U^+(t) - U^-(t) = g(t) \quad \text{on } \Gamma, \tag{9}$$

where $g(t)$ is an H-continuous function.

Theorem 1. *For the solvability of problem (9) it is necessary and sufficient that*

$$\operatorname{Im}\int_\Gamma g(t)V_j^0(t)\,dt = 0 \quad (j = 1, 2, \dots, k_1), \tag{10}$$

where V_j^0 ($j = 1, 2, \dots, k_1$) is a basis of generalized covariants of the first kind belonging to equation (5).

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* The assumptions concerning the smoothness and homological properties of Γ and T^+ are the same as in (7).

The necessity of condition (10) is obvious. To prove sufficiency, we seek a solution in the form

$$U^{\pm}(z) = \frac{1}{2\pi i}\int_{\Gamma} \varphi(\tau)\Omega_1^{\pm}(\tau, z)\, d\tau - \overline{\varphi(\tau)}\, \Omega_2^{\pm}(\tau, z)\, d\bar{\tau}. \tag{11}$$

We arrive at the Fredholm equation

$$\varphi(t) + \frac{1}{2\pi i}\int_{\Gamma} \varphi(\tau)\left[\Omega_1^{+}(\tau, t) - \Omega_1^{-}(\tau, t)\right] d\tau - \overline{\varphi(\tau)}\left[\Omega_2^{+}(\tau, t) - \Omega_2^{-}(\tau, t)\right] d\bar{\tau} = g(t). \tag{12}$$

Introduce the scalar product*

$$(\varphi, \psi) = \operatorname{Re}\int_{\Gamma} \varphi(\tau)\overline{\psi(\tau)}\, ds_{\tau}, \tag{13}$$

where ds_{τ} is the element of length of the contour in an arbitrary metric. The equation adjoint to (12) in the metric (13) will be the covariant equation

$$\omega(t) + \frac{1}{2\pi i}\int_{\Gamma} \omega(\tau)\left[\Omega_1^{+}(t, \tau) - \Omega_1^{-}(t, \tau)\right] d\tau + \overline{\omega(\tau)}\left[\overline{\Omega_2^{+}(t, \tau)} - \overline{\Omega_2^{-}(t, \tau)}\right] d\bar{\tau} = 0. \tag{14}$$

The solutions of equation (14) are the covariants $iV_j^0(t)$ $(j = 1, 2, \dots, k_1)$, and only these; whence the assertion of the theorem follows.

Consider the Riemann problem

$$U^{+}(t) = G(t)U^{-}(t) + g(t) \quad \text{on } \Gamma \tag{15}$$

for equation (1). We call the adjoint problem

$$V^{+}(t) = \frac{1}{G(t)}V^{-}(t) \tag{16}$$

for equation (5).

From the reasoning given in the proof of Theorem 1, it follows that the solution of problem (15) is representable in the form

$$U^{\pm}(z) = \frac{1}{2\pi i}\int_{\Gamma} \varphi(\tau)\Omega_1^{\pm}(\tau, z)\, d\tau - \overline{\varphi(\tau)}\, \Omega_2^{\pm}(\tau, z)\, d\bar{\tau} + \sum_{j=1}^{k_0''} x_j U_j^0(z), \tag{17}$$

where $U_j^0(z)$ $(j = 1, \dots, k_0'')$ is a basis of the space of those generalized constants which are not representable in the form (11) $(0 \le k_0'' \le 2)$. We arrive at the singular equation

$$\frac{1+G(t)}{2}\varphi(t)+\frac{1}{2\pi i}\int_\Gamma \varphi(\tau)\left[\Omega_1^+(\tau,t)-G(t)\Omega_1^-(\tau,t)\right]d\tau-\overline{\varphi(\tau)}\left[\Omega_2^+(\tau,t)-G(t)\Omega_2^-(\tau,t)\right]d\bar{\tau}=g(t)+\sum_{j=1}^{k_0''}x_j\,[G(t)- \tag{18}$$

of index $\chi=\operatorname{ind}_\Gamma G$. The equation adjoint to (18) in the metric (13) will be

$$\frac{1+\overline{G(t)}}{2}\omega(t)+\frac{1}{2\pi i}\int_\Gamma \omega(\tau)\left[\Omega_1^+(t,\tau)-\overline{G(\tau)}\Omega_1^-(t,\tau)\right]d\tau+\overline{\omega(\tau)}\left[\overline{\Omega_2^+(t,\tau)}-\overline{G(\tau)}\,\overline{\Omega_2^-(t,\tau)}\right]d\bar{\tau}=0. \tag{19}$$

The solutions of this equation are the covariants $iV_j^+(t)$ $(j=1,2,\dots,l')$, where $V_j^+(t)$ are solutions of the adjoint problem (16), and only these.

* The idea of introducing the metric (13) was borrowed by us from B. V. Boyarskii (^{3}).

Let $\eta(t)$ be a solution of the homogeneous equation (18) to which, by formulas (17), there corresponds the zero solution of problem (15) ($g(t)\equiv 0$). Then it will also be a solution of the homogeneous equation (12). The number of such functions is equal to k_1-k_0', where k_0' is the number of generalized constants representable in the form (11), $k_0'+k_0''=k_0$. Therefore the number l of solutions of the homogeneous problem (15) and the number $\tilde{l}$ of solutions of the homogeneous equation (18) are related by

$$l=\tilde{l}-(k_1-k_0')+k_0''=\tilde{l}-k_1+k_0.$$

It can be shown that $k_1-k_0=2p-2$. Taking into account that the number l' of solutions of problem (16) and the number $\tilde{l}'$ of solutions of equation (19) are related by $l'=\tilde{l}'$, and $\tilde{l}-\tilde{l}'=2\varkappa$, we obtain the following result:

Theorem 2. *The difference between the number of solutions l of the homogeneous problem (15) and l' of the homogeneous problem (16) is equal to*

$$l-l'=2\varkappa-2p+2.$$

In particular, for $\varkappa\geq p$ the problem is always solvable.

Theorem 3. *For the solvability of the nonhomogeneous problem (15), it is necessary and sufficient that*

$$\operatorname{Im}\int_\Gamma g(t)V_j^+(t)\,dt=0 \qquad (j=1,2,\dots,l'), \tag{20}$$

where V_j^+ $(j=1,2,\dots,l')$ is a basis of the space of solutions of problem (16).

The Hilbert boundary-value problem

$$\operatorname{Re}[(a-ib)U(t)]=c(t) \tag{21}$$

on a surface T of genus h with $m+1$ boundary contours, by the method of paper (6), is reduced to the Riemann problem

$$U^+(t) = -\frac{a(t)+ib(t)}{a(t)-ib(t)}\,U^-(t) + \frac{2c(t)}{a(t)-ib(t)} \tag{22}$$

on the surface R—the double of T; moreover the coefficient of equation (1) for the domain T is determined by the formula

$$B(z) = \overline{B(\tilde{z})},$$

where $P[z] \in \tilde{T}$ and $\tilde{z}$ is a local parameter of the point symmetric to P. The problem conjugate to (21) is called the covariant problem

$$\mathrm{Re}[(a+ib)V(t)] = 0 \tag{23}$$

for the conjugate equation.

Theorem 4. *The difference between the number of solutions l of the homogeneous problem (21) and l' of problem (23) is equal to*

$$l - l' = 4\varkappa - 4h - 2m + z,$$

where $\varkappa = \mathrm{ind}(a+ib)$. For the solvability of the nonhomogeneous problem (21), it is necessary and sufficient that

$$\int_\Gamma c(t)[a(t)+ib(t)]V_j(t)\,dt = 0 \qquad (j = 1, \dots, l'),$$

where $V_j(t)$ is a complete system of solutions of the conjugate problem. For $h=0$, the known results of I. N. Vekua (1) follow from this.

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Received
24 II 1961

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* The nonsolvability of the problem for $\varkappa < 0$ follows from the argument principle.