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Abstract

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MATHEMATICS

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On Direct and Inverse Embedding Theorems for Some Spaces of Abstract Set Functions

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Let $\Phi(E)$ be an abstract additive set function, defined on all Lebesgue-measurable subsets E of a bounded domain Ω of Euclidean space R_n, with values in a Banach space X; denote by $\bar{X}$ the space conjugate to X. We shall say that the function $\Phi(E)$ belongs to the space $\Phi_p(X,\Omega)$, introduced by S. L. Sobolev (^{1}), if

$$\|\Phi(E)\|_{\Phi_p(X,\Omega)} = \sup_{\tilde{\omega}} \frac{\left\|\int_\Omega \tilde{\omega}(x)\, d_x\Phi(E)\right\|_X}{\|\tilde{\omega}\|_{L_{p'}(\Omega)}} < \infty,$$

where $\tilde{\omega}(x)$ is a real measurable function taking only a finite number of nonzero values, $\frac{1}{p}+\frac{1}{p'}=1$, $p>1$. Denote by $\tilde{\Psi}_p(X,\Omega)$ (^{1}) the totality of all abstract set functions from Φ_p that are absolutely continuous in the metric $\| \ \|_{\Phi_p}$; the totality of all abstract functions from Φ_p continuous with respect to shifts in the metric $\| \ \|_{\Phi_p}$ will be denoted by $\Psi_p(X,\Omega)$ (^{1}).

It is known that every function $\Phi(E)$ from $\Phi_p(X,\Omega)$, $p>1$, is absolutely continuous in the metric of the space X (^{1,2}). Indeed, let $\tilde{\omega}=\chi_e$, where $\chi_e(x)$ is the characteristic function of the set e; then

$$\|\Phi(e)\|_X = (\operatorname{mes} e)^{1/p'} \frac{\|\Phi(e)\|_X}{(\operatorname{mes} e)^{1/p'}} =$$

$$= \frac{\left\|\int_\Omega \chi_e(x)\, d_x\Phi(E)\right\|_X}{(\operatorname{mes} e)^{1/p'}} (\operatorname{mes} e)^{1/p'} \leq (\operatorname{mes} e)^{1/p'} \|\Phi(E)\|_{\Phi_p}.$$

Consequently, for any $f\in\bar{X}$ the real-valued set function $f\Phi(E)$ is absolutely continuous and therefore can be represented in the form

$$f\Phi(E) = \int_E \varphi_f(x)\, dx.$$

Lemma 1. Let $\Phi(E) \in \Phi_p(X, \Omega)$, $p > 1$. Then

$$\|\Phi(E)\|_{\Phi_p(X,\Omega)} = \sup_{\tilde{\omega}} \frac{\left\| \int_\Omega \tilde{\omega}\, d_x \Phi(E) \right\|_X}{\|\tilde{\omega}\|_{L_{p'}}} = \sup_{\|f\| \leq 1} \|\varphi_f\|_{L_p(\Omega)}.$$

For $p = 1$ an analogous assertion holds for functions $\Phi(E)$ absolutely continuous in the metric X, forming the space $\bar{\Phi}_1$, defined in (^{2}). The lemma remains valid if $p > 1$ and $\operatorname{mes} \Omega = \infty$, with the corresponding definition of the space $\Phi_p(X, \Omega)$.

Corollary. Let $\Phi(E) \in \Phi_p(X, \Omega)$. Then the correspondence $f \to \varphi_f(x)$ is a linear continuous operator Φ acting from $\overline{X}$ into $L_p(\Omega)$, and

$$\|\Phi(E)\|_{\Phi_p(X,\Omega)} = \|\Phi\|.$$

In this case we shall say that the operator Φ is generated by the function $\Phi(E)$. We denote by $\mathfrak{M}_\Phi$ the set $\{\varphi_f(x), \|f\| \leqslant 1\}$; $\mathfrak{M}_\Phi \subset L_p(\Omega)$.

Definition 1. By $(B_1 \to B_2; \mathfrak{M})$ we denote the space of all linear operators acting from B_1 into B_2 and possessing a certain property $\mathfrak{M}$.

Definition 2. We shall call an operator

$$\Phi \in (\overline{X} \to L_p(\Omega);\ \text{cont.})$$

absolutely continuous if its norm is arbitrarily small when $\operatorname{mes} \Omega \to 0$, i.e., if $\mathfrak{M}_\Phi$ is a family of functions with uniformly absolutely continuous norms ((^{3}), p. 117).

Theorem 1. A function $\Phi(E)$ from $\Phi_p(X, \Omega)$ generates an operator

$$\Phi \in (\overline{X} \to L_p(\Omega);\ \text{cont.}),$$

a function $\Phi(E)$ from $\widetilde{\Psi}_p(X, \Omega)$ generates an operator

$$\Phi \in (\overline{X} \to L_p(\Omega);\ \text{abs. cont.}),$$

and a function $\Phi(E)$ from $\Psi_p(X, \Omega)$ generates an operator

$$\Phi \in (\overline{X} \to L_p(\Omega);\ \text{completely cont.}).$$

Moreover

$$\|\Phi(E)\|_{\Phi_p(X,\Omega)} = \|\Phi\|.$$

Thus, $\Phi(E) \in \Phi_p$ (respectively $\Phi(E) \in \widetilde{\Psi}_p$, respectively $\Phi(E) \in \Psi_p$) if and only if $\mathfrak{M}_\Phi$ is a family of functions bounded (respectively with uniformly absolutely continuous norms, respectively compact) in $L_p(\Omega)$.

Hence, and from the compactness criterion of M. A. Krasnosel' skii (^{3}), there follows immediately

Theorem 2.

$$\widetilde{\Psi}_p = \Psi_1 \cap \widetilde{\Psi}_p, \qquad p > 1.$$

A natural generalization of Krasnosel’ skii’ s criterion is

Theorem 3. Let $\mathfrak{M} \subset \widetilde{\Psi}_p(X, \Omega)$. Then $\mathfrak{M}$ is compact in $\widetilde{\Psi}_p(X, \Omega)$ if and only if $\mathfrak{M}$ is a family of functions uniformly absolutely continuous in the metric $\| \ \|_{\Phi_p}$, whose set is compact in $\Phi_1(X, \Omega)$.

Now let X be the space conjugate to some Banach space B, i.e. $X = \overline{B}$. As above, one can show that if $\Phi(E) \in \Phi_p(X, \Omega)$, then for any $g \in B$ the scalar function $\Phi(E)g$ is absolutely continuous and therefore representable in the form

$$\Phi(E)g = \int_E \varphi_g(x)\, dx.$$

Lemma 2.

$$\|\Phi(E)\|_{\Phi_p(X,\Omega)} = \sup_{\|g\|_B \leqslant 1} \|\varphi_g\|_{L_p(\Omega)}.$$

Thus, a function $\Phi(E) \in \Phi_p(X, \Omega)$ generates an operator

$$\Phi \in (B \to L_p(\Omega);\ \text{cont.}),$$

and moreover

$$\|\Phi(E)\|_{\Phi_p(X,\Omega)} = \|\Phi\|.$$

Conversely, let

$$\Phi \in (B \to L_p(\Omega);\ \text{cont.}).$$

Then

$$\left|\int_E \Phi(g)\, dx\right| \leqslant \|\Phi\| (\operatorname{mes} \Omega)^{1/p'} \|g\|,$$

i.e.

$$\int_E \Phi(g)\, dx$$

is a continuous linear functional on B. Taking

$$\int_E \Phi(g)\, dx = \Phi(E)g,$$

we are convinced that $\Phi(E)$ is an additive abstract function of sets, and

$$\|\Phi(E)\|_{\Phi_p(X,\Omega)} = \|\Phi\|.$$

Theorem 4. If $X = \overline{B}$, then

$$\Phi_p(X, \Omega) \leftrightarrow (B \to L_p(\Omega);\ \text{cont.}),$$

$$\widetilde{\Psi}_p(X,\Omega) \leftrightarrow (B \to L_p(\Omega);\ \text{abs. cont.}),$$

$$\Psi_p(X,\Omega) \leftrightarrow (B \to L_p(\Omega);\ \text{completely cont.}),$$

where the sign $\leftrightarrow$ denotes an isometric and isomorphic correspondence defined by the formula

$$\int_E \Phi(g)\,dx = \Phi(E)g. \tag{*}$$

Example 1. Let $X = R_m$; then

$$\Phi_p(R_m,\Omega) = \widetilde{\Psi}_p(R_m,\Omega) = \Psi_p(R_m,\Omega).$$

Example 2. Let $X = L_2(\Omega)$, $p = 2$, $\Phi = I$, where I is the identity operator. From (*) it is seen that I is generated by the function $\Phi(E) = \chi_E(x)$, $\chi_E(x)$ being the characteristic function of the set E. It is easy to see that $\Phi(E) \in \Phi_2(L_2(\Omega),\Omega) \setminus \Psi_2(L_2(\Omega);\Omega)$.

Example 3. Let $X = L_2(\Omega)$, $1 < p < 2$, $\Phi = V$, where V is the operator of embedding $L_2(\Omega)$ into $L_p(\Omega)$. Then $\Phi(E) = \chi_E$ and $\Phi(E) \in \Psi_p(L_2(\Omega),\Omega) \setminus \widetilde{\Psi}_p(L_2(\Omega),\Omega)$

Let

$$\Psi(E) = \frac{\partial^l \Phi(E)}{\partial x_1^{l_1} \ldots \partial x_n^{l_n}}$$

be the generalized, in the sense of S. L. Sobolev, derivative of the function $\Phi(E)$ (1). Denote by $\Phi_p^{(l)}(X,\Omega)$ the collection of those and only those functions $\Phi(E)$ from $\overline{\Phi}_1(X,\Omega)$ all of whose generalized derivatives of order l belong to $\Phi_p(X,\Omega)$. In $\Phi_p^{(l)}(X,\Omega)$ introduce the norm

$$\|\Phi(E)\|_{\Phi_p^{(l)}(X,\Omega)} = \|\Phi(E)\|_{\Phi_1(X,\Omega)} + \sum_{l_1+\ldots+l_n=l} \left\| \frac{\partial^l \Phi(E)}{\partial x_1^{l_1} \ldots \partial x_n^{l_n}} \right\|_{\Phi_p(X,\Omega)}.$$

From the fact that $\Phi(E) \in \Phi_p^{(l)}$ it follows that

$$\Phi(E)g = \int_E \varphi_g(x)\,dx, \qquad \frac{\partial^l \Phi(E)}{\partial x_1^{l_1} \ldots \partial x_n^{l_n}} g = \int_E \frac{\partial^l}{\partial x_1^{l_1} \ldots \partial x_n^{l_n}} \varphi_g(x)\,dx.$$

From Lemma 2 there follows the equivalence of the norm $\|\Phi(E)\|_{\Phi_p^{(l)}}$ to the norm

$$^{(1)}\|\Phi(E)\|_{\Phi_p^{(l)}(X,\Omega)} = \sup_{\|g\|_B \le 1} \|\varphi_g\|_{W_p^{(l)}},$$

which we shall call the canonical norm. The space $\Psi_p^{(l)}(X,\Omega)$ (1,2) of all functions $\Phi(E)$ from $\overline{\Phi}_1(X,\Omega)$ having all generalized derivatives of order l, continuous with respect to shifts in the metric $\| \ \|_{\Phi_p}$, forms a subspace in $\Phi_p^{(l)}(X,\Omega)$.

Now let $\lambda > 0$, λ noninteger, and $\lambda = \bar{\lambda} + \alpha$, $\bar{\lambda}$ integer, $0 < \alpha < 1$, $1 < p < \infty$. We shall say that $\Phi(E) \in \Phi_p^{(\lambda)}(X, \Omega)$ if $\Phi(E) \in \Phi_p^{(\bar{\lambda})}(X, \Omega)$ and, for any generalized derivative $D^{(\bar{\lambda})}$ of order $\bar{\lambda}$ of the function $\Phi(E)$, the inequality

$$\sup_{\|g\|_B \le 1} \int_\Omega \int_\Omega \frac{\left|D^{(\bar{\lambda})}\varphi_g(x) - D^{(\bar{\lambda})}\varphi_g(y)\right|^p}{|x-y|^{n+p\alpha}}\, dx\, dy < \infty$$

holds.

The norm in $\Phi_p^{(\lambda)}$, defined by the equality

$$\|\Phi(E)\|_{\Phi_p^{(\lambda)}} = \|\Phi(E)\|_{\Phi_p^{(\bar{\lambda})}} +$$

$$+ \sum_{\lambda_1 + \ldots + \lambda_n = \bar{\lambda}} \sup_{\|g\|_B \le 1} \left(\int_\Omega \int_\Omega \frac{\left|D^{(\bar{\lambda})}\varphi_g(x) - D^{(\bar{\lambda})}\varphi_g(y)\right|^p}{|x-y|^{n+p\alpha}}\, dx\, dy \right)^{1/p},$$

is equivalent to the canonical norm

$$^{(1)}\|\Phi(E)\|_{\Phi_p^{(\lambda)}} = \sup_{\|g\|_B \le 1} \|\varphi_g\|_{W_p^{(\lambda)}},$$

where $W_p^{(\lambda)}(\Omega)$ is the space considered by L. N. Slobodetskii (4), O. V. Besov (5), and others. The formally narrower class

$$\overline{\Phi}_p^{(\lambda)}(X, \Omega) = \Psi_p^{(\bar{\lambda})} \cap \Phi_p^{(\lambda)}$$

in fact coincides with $\Phi_p^{(\lambda)}$. Let us also define the space $\Psi_p^{(\lambda)}(X, \Omega)$. $\Phi(E) \in \Psi_p^{(\lambda)}$ if $\Phi(E) \in \Phi_p^{(\lambda)}$ and

$$\lim_{h \to 0} \|\Phi_h(E) - \Phi(E)\|_{\Phi_p^{(\lambda)}} = 0,$$

where $\Phi_h(E)$ is the mean function (1).

Theorem 5. Let $X = \overline{B}$, $1 < p < \infty$, $r > 0$, and let the norms in the spaces $\Phi_p^{(r)}$ and $\Psi_p^{(r)}$ be canonical. Then

$$\Phi_p^{(r)}(X, \Omega) \leftrightarrow (B \to W_p^{(r)}(\Omega);\ \text{cont.}), \qquad \Psi_p^{(r)}(X, \Omega) \leftrightarrow (B \to W_p^{(r)}(\Omega);\ \text{completely cont.}),$$

where the symbol $\leftrightarrow$ denotes an isometric and isomorphic correspondence defined by formula $(*)$.

The correspondences obtained in Theorem 5 make it possible to prove, for the spaces $\Phi_p^{(r)}$ and $\Psi_p^{(r)}$, theorems analogous to the known embedding and extension theorems for the spaces $W_p^{(r)}$. Let us prove, for example, the following theorem (see, for example, [6], p. 111):

Let $X = \overline{B}$; $0 \le k = l - \frac{n}{p} + \frac{m}{p'}$; $1 < p < p' < \infty$; Ω be a domain all of whose points are attainable by means of a fixed cone; then $\Phi_p^{(l)}(X, \Omega_n)$ is embedded in $\Phi_{p'}^{(k)}(X, \Omega_m)$.

Indeed, $\Phi(E) \in \Phi_p^{(l)}(X, \Omega)$ generates an operator $\Phi \in (B \to W_p^{(l)}(\Omega_n),\ \text{cont.})$. In turn, the operator $W = V\Phi$, where V is the embedding operator of $W_p^{(l)}(\Omega_n)$ into $W_{p'}^{(k)}(\Omega_m)$, generates a function $\widetilde{\Phi}(I) \in \Phi_{p'}^{(k)}(X, \Omega_m)$, which is the trace of the function $\Phi(E)$, and moreover

$$\|\widetilde{\Phi}(I)\|_{\Phi_{p'}^{(k)}(X,\Omega_m)} = \|W\| \leq\leq \|V\| \cdot \|\Phi\| = \|V\| \cdot \|\Phi(E)\|_{\Phi_q^{(l)}(X,\Omega_n)}.$$

The corresponding theorems on extension of abstract functions of sets are proved analogously.

Let us note, however, that in the case when $X = \overline{B}$ is an infinite-dimensional Banach space, the following two features occur:

1) The embedding operator is not completely continuous, although estimates of the type of the estimates of V. I. Kondrashev [7] do hold. This is connected with the fact that the compactness criterion in $L_p(\Omega)$, $p > 1$, due to M. Riesz [8], cannot be generalized to the space $\Psi_p(X, \Omega)$.

2) For $p > 1$, $lp > n$, l natural, every function $\Phi(E) \in \Phi_p^{(l)}(X, \Omega)$ is representable in the form of an indefinite integral of I. M. Gelfand [9] of a weakly continuous abstract function of points $x \in \Omega$, i.e.

$$\Phi(E) = \int_E \varphi(x)\, dx,$$

and

$$\sup_{\|g\|B \leq 1} \max_{x \in \Omega} |\varphi(x) g| \leq c \|\Phi(E)\|_{\Phi_p^{(l)}}.$$

Let X be an arbitrary Banach space. The application of Lemma 1 and of the apparatus of mean functions makes it possible to obtain in a simple way, for the spaces $\Psi_p^{(r)}$, analogues of some known embedding and extension theorems [10].

Finally, we note that a number of embedding theorems in which the dimension of the domain Ω remains unchanged (of the type of the Gagliardo-Nirenberg theorems [11,12], etc.) also hold for the spaces $\Phi_p^{(l)}(X, \Omega)$. In this case X may also be an arbitrary Banach space.

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Note: Figure translations are in progress. See original paper for figures.