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Abstract

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Yu. OREVKOV

ON TOTALLY BOUNDED METRIC SPACES

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Let P be a metric space; uP its bicompact extension corresponding to the metric proximity ([1]). Yu. M. Smirnov ([2]) characterized completeness of a space in a given metric through the properties of the remainder of the bicompact extension $uP \setminus P$. The property of a metric space P of being totally bounded, which, like completeness, is preserved under uniformly continuous mappings, can also be described with the aid of the remainder $uP \setminus P$. This is done in the present note.

Theorem 1. *In order that a metric space P be totally bounded, it is necessary and sufficient that $uP \setminus P$ be hereditarily normal.*

Necessity. Since in the case when P is totally bounded the extension uP is a compactum, $uP \setminus P$ is evidently hereditarily normal.

Sufficiency. Let P be a metric space that is not totally bounded. Then there exists a countable set $N = \{x_i\} \subset P$ such that for any two distinct points x_i and x_j one always has $\rho(x_i, x_j) \geq \eta > 0$. Consequently, the set N is uniformly equivalent to the set of natural numbers of the real line. The bicompact extension corresponding to the proximity space of the natural sequence is its Čech extension, and, since $N \subset P$, $P[N] = N$, it follows that $\beta N = uN = uP[N]$. Therefore, in order to show that the difference $uP \setminus P$ is not hereditarily normal, it suffices to prove the following assertion:

Lemma. *$\beta N \setminus N$ is not hereditarily normal.*

Proof. Let I be the half-interval $[0, 1)$, with the neighborhoods of any point $x \in I$ being half-intervals of the form $[x, y)$. Consider the space $R = I \times I$. As is known ([3]), R is a completely regular, but not normal, space having a countable everywhere dense subset $\{a_i\}$. Let γR be some bicompact extension of the space R. We construct a bicompact extension $\tau N = \gamma R \cup N$ of the countable discrete space N, where the topology is defined as follows: a neighborhood of a point from N is the point itself; a neighborhood of a point from γR is an ordinary neighborhood in γR with the addition of those natural numbers which serve as indices of the points from $\{a_i\}$ falling into this neighborhood. The bicompact extension of the natural sequence thus obtained has a remainder that is not hereditarily normal, since R is its subspace. In view of the property* of the Čech extension, we have $f(\beta N \setminus N) = \tau N \setminus N$. Hence f is a closed continuous mapping of the space $f^{-1}(R)$ onto R, and therefore $f^{-1}(R)$ is a completely regular but not normal subspace of $\beta N \setminus N$. The lemma and, consequently, Theorem 1 are proved.

The lemma makes it possible to prove the following assertion:

Theorem 2. *Every completely regular non-pseudocompact space R has a Čech remainder which is not hereditarily normal.*

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* For otherwise, for some point $y \in N$ (which is open in τN), we would have that $f^{-1}y$ intersects $\beta N \setminus N$ and, being an open set, contains infinitely many points from N, which is impossible.

Proof. In R there exists a continuous unbounded function f and, consequently, a countable subset $A = \{a_i\}$ such that $|f(a_i) - f(a_j)| > 1$ for $i \neq j$. Let φ be any real function defined on A. We shall prove that it can be extended continuously to all of R. To this end consider neighborhoods Oa_i on which the oscillation of the function f does not exceed 1/3. Since R is completely regular, there exists a function ψ_i, continuous on R, such that $\psi_i(a_i) = \varphi(a_i)$ and $\psi_i(x) = 0$ for $x \in R \setminus Oa_i$, with $0 \le \psi_i(x) \le \varphi(a_i)$ at points $x \in Oa_i$ when $\varphi(a_i) \ge 0$, and $0 \ge \psi_i(x) \ge \varphi(a_i)$ at points $x \in Oa_i$ when $\varphi(a_i) \le 0$. It is clear that

$$\text{the function } \psi(x) = \sum_{i=1}^{\infty} \psi_i(x)$$

is continuous on R and that $\psi(a_i) = \varphi(a_i)$.

Consequently, $\beta R[A] = \beta A$. Hence, by the lemma, the absence of hereditary normality in $\beta R \setminus R$ follows. In Theorem 1 the condition of hereditary normality can be replaced by any condition from which hereditary normality follows and which holds for subsets of compacta. Therefore the following is true:

Theorem 3. Each of the following conditions is necessary and sufficient for the metric space P to be totally bounded:

a) $uP \setminus P$ is perfectly normal (every open set is of type F_σ);

b) $uP \setminus P$ is hereditarily finally compact (Lindelöf);

c) $uP \setminus P$ is hereditarily strongly paracompact;

d) $uP \setminus P$ is hereditarily paracompact;

e) $uP \setminus P$ has a countable base.

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Note: Figure translations are in progress. See original paper for figures.