N. V. KUZNETSOV

1962

SovietRxiv

Abstract

Full Text

N. V. KUZNETSOV

A GENERALIZATION OF A THEOREM OF V. A. AMBARTSUMIAN

(Presented by Academician I. G. Petrovsky, 7 V 1962)

In this article we give a generalization of a theorem of V. A. Ambartsumian concerning the inverse Sturm–Liouville problem on a finite interval. In the paper ([1]) V. A. Ambartsumian considered the equation

$$\varphi''(x) + (\lambda - q(x))\varphi(x) = 0, \qquad 0 \le x \le \pi, \qquad \varphi'(0) = \varphi'(\pi) = 0, \quad (1)$$

and proved that if the spectrum of this problem coincides with the spectrum of the unperturbed equation

$$\varphi''(x) + \lambda\varphi(x) = 0, \qquad \varphi'(0) = \varphi'(\pi) = 0, \quad (2)$$

then $q(x) \equiv 0$. We extend this result to the two-dimensional and three-dimensional cases.

For definiteness, in what follows we shall everywhere consider the three-dimensional case. The reasoning for the two-dimensional case is completely analogous. Let G be a finite domain with sufficiently smooth boundary Γ, and in G let the equation be given

$$\Delta\varphi + (\lambda - V(x))\varphi = 0, \qquad \left.\frac{\partial\varphi}{\partial n}\right|_{\Gamma} = 0. \quad (3)$$

As is known, its eigenvalues form a discrete sequence μ_n. We denote the eigenvalues of the unperturbed equation by λ_n. We formulate the main result of the paper.

Theorem 1. *Suppose the series* $\sum_{n=1}^{\infty}(\mu_n - \lambda_n)$ *converges and the first eigenvalue of problem (3) coincides with the first eigenvalue of the unperturbed equation, i.e. of the equation*

$$\Delta\varphi + \lambda\varphi = 0, \qquad \left.\frac{\partial\varphi}{\partial n}\right|_{\Gamma} = 0. \quad (3')$$

Then $V(x)$ *is identically equal to zero.*

Let us outline the main idea of the proof. First of all, we prove that a necessary condition for the convergence of the series $\sum\limits_{n=1}^{\infty}(\mu_n - \lambda_n)$ is the equality

$$\int_G V\,dx = 0. \tag{4}$$

After this it is no longer difficult to prove that, under the conditions

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\Gamma} = 0, \qquad \mu_1 = \lambda_1 = 0,$$

$V(x)$ must be identically zero. Indeed, the first eigenvalue is the absolute minimum of the functional

$$D(\varphi) = \int_G (\nabla\varphi)^2\,dx + \int_G V(x)\varphi^2\,dx.$$

on all normalized functions satisfying the boundary conditions. Since $\int_G V\,dx = 0$, on the constant $D(\varphi)$ vanishes, and, consequently, under the conditions

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\Gamma} = 0$$

the constant is an eigenfunction of problem (3), and directly from the equation we conclude that $V(x) \equiv 0$.

We shall derive condition (4) from the following theorem:

Theorem 2. *Let T be a self-adjoint semibounded-below operator having a completely continuous inverse of Hilbert-Schmidt type. Let, furthermore, V be a bounded symmetric operator; λ_n, μ_n are the eigenvalues of $T, T+V$, respectively. Then, if the series $\sum_{n=1}^{\infty}(\mu_n - \lambda_n)$ converges, then*

$$\sum_{n=1}^{\infty}(\mu_n - \lambda_n) = \lim_{\rho\to\infty} \rho^2 \left\{\mathrm{Sp}\,(T_\rho^{-1} V T_\rho^{-1}) - \mathrm{Sp}\,(T_\rho^{-1}(V T_\rho^{-1})^2)\right\}, \tag{5}$$

where the notation $T_\rho = T + \rho E$ has been introduced.

This theorem is a generalization of a result due to Halberg and Kramer ([2]). They proved an analogous formula for the case when T_ρ^{-1} has a trace.

We shall give the proof of Theorem 2 at the end of the article, and now pass to the proof of the necessity of condition (4).

For the eigenvalues of the Laplace equation in a finite domain one has the asymptotic formula

$$\lambda_n = \left(\frac{6\pi}{\mathrm{mes}\,G}\right)^{2/3} n^{2/3} + o(n^{2/3}).$$

Consequently, the series $\sum_{n=1}^{\infty}\frac{1}{(\lambda_n+\rho)^2}$ converges for sufficiently large ρ, and if

$$\sum_{n=1}^{\infty}(\mu_n-\lambda_n)<\infty,$$

then the conditions of Theorem 2 are fulfilled, and we may write

$$\sum_{n=1}^{\infty}(\mu_n-\lambda_n)=\lim_{\rho\to\infty}\rho^2\left\{\mathrm{Sp}\,(T_\rho^{-1}VT_\rho^{-1})-\mathrm{Sp}\,(T_\rho^{-1}(VT_\rho^{-1})^2)\right\}. \tag{6}$$

We write the traces entering the right-hand side of (6) in the basis of eigenfunctions φ_n of the unperturbed equation (3′), which gives

$$\sum_{n=1}^{\infty}(\mu_n-\lambda_n)=\lim_{\rho\to\infty}\rho^2\left\{\sum_{n=1}^{\infty}\frac{(V\varphi_n,\varphi_n)}{(\lambda_n+\rho)^2}-\sum_{n=1}^{\infty}\frac{(VT_\rho^{-1}V\varphi_n,\varphi_n)}{(\lambda_n+\rho)^2}\right\}. \tag{7}$$

Let us estimate each of the terms on the right-hand side of (7) for finite ρ. We note the obvious inequality

$$\|T_\rho^{-1}\|\le\frac{1}{\lambda_1+\rho},$$

where λ_1 is the first eigenvalue of T. Consequently, the second term does not exceed

$$\frac{\|V\|^2}{\lambda_1+\rho}\sum_{n=1}^{\infty}\frac{1}{(\lambda_n+\rho)^2}.$$

Next, we use Titchmarsh' s estimate ([3])

$$\sum_{n=1}^{\infty}\frac{\varphi_n^2(x)}{(\lambda_n+\rho)^2}=\frac{1}{8\pi\sqrt{\rho}}-F_\rho(x), \tag{8}$$

where F_ρ preserves its sign inside G and $F_\rho=O\left(\frac{1}{\sqrt{\rho}}e^{-a\sqrt{\rho}}\right)$. Here a is the distance from the point x to the boundary Γ. Let us also note that, by Dini' s theorem (4), the series (8), as one converging to a continuous sum of a series of positive functions, converges uniformly in x. Therefore one may write

$$\rho^2\sum_{n=1}^{\infty}\frac{(V,\varphi_n^2)}{(\lambda_n+\rho)^2}=\rho^2\left(V,\sum_{n=1}^{\infty}\frac{\varphi_n^2}{(\lambda_n+\rho)^2}\right),$$

which gives

$$\rho^2\,\mathrm{Sp}(T_\rho^{-1}VT_\rho^{-1})=\frac{\rho^{3/2}}{8\pi}(V,1)-\rho^2\int_G F_\rho V\,dx. \tag{9}$$

Since F_ρ preserves its sign inside G, we have

$$\left|\int_G F_\rho(x)V(x)\,dx\right| \le \max_G |V| \left|\int_G F_\rho(x)\,dx\right|.$$

Cut out from G the domain G_h, all points of which are at distance at least h from the boundary Γ, and suppose that, as $h \to 0$, $\operatorname{mes}(G - G_h) = O(h)$. Then, since $F_\rho = O\left(\frac{1}{\sqrt{\rho}} e^{-a\sqrt{\rho}}\right)$ and V is bounded, h can be chosen so that

$$\rho^2 \max_G |V| \left|\int_{G-G_h} F_\rho\,dx\right| \le \frac{\varepsilon\rho^{3/2}}{8\pi}|(V,1)| \qquad ((V,1) \ne 0).$$

for any prescribed $\varepsilon > 0$. The integral over G_h gives an exponentially small contribution and may be discarded. Finally:

$$\rho^2 \operatorname{Sp}(T_\rho^{-1} V T_\rho^{-1}) = \frac{1+\vartheta\varepsilon}{8\pi}\rho^{3/2}(V,1), \qquad |\vartheta| < 1. \tag{10}$$

Putting here $V = 1$, we obtain the following equality for the trace of the operator $\rho^2 T_\rho^{-2}$:

$$\rho^2 \operatorname{Sp} T_\rho^{-2} = \rho^2 \sum_{n=1}^{\infty} \frac{1}{(\lambda_n + \rho)^2} = \frac{1+\eta\varepsilon}{8\pi}\rho^{3/2} \operatorname{mes} G, \qquad |\eta| < 1. \tag{11}$$

Combining (7), (10), and (11), we obtain

$$\sum_{n=1}^{\infty}(\mu_n - \lambda_n) = \lim_{\rho\to\infty} \rho^2 \left\{\operatorname{Sp}(T_\rho^{-1} V T_\rho^{-1}) - \operatorname{Sp}(T_\rho^{-1}(V T_\rho^{-1})^2)\right\} =$$

$$= \lim_{\rho\to\infty} \left(\frac{1+\vartheta\varepsilon}{8\pi}\rho^{3/2}(V,1) - O(\rho^{1/2})\right). \tag{12}$$

Consequently, the series $\sum_{n=1}^{\infty}(\mu_n - \lambda_n)$ can converge only in the case where $(V,1) = 0$. The theorem is proved.

We now prove Theorem 2. First of all, the equality holds

$$\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n + \rho} - \frac{1}{\mu_n + \rho}\right) = \operatorname{Sp}\{T_\rho^{-1} - (T_\rho + V)^{-1}\}. \tag{13}$$

Indeed, since

$$T_\rho^{-1} - (T_\rho + V)^{-1} = T_\rho^{-1}(E + V T_\rho^{-1})^{-1} V T_\rho^{-1},$$

where the operator $(E+VT_\rho^{-1})^{-1}V$ is bounded, while T_ρ^{-1} is of Hilbert-Schmidt type, it follows that $\mathrm{Sp}(T_\rho^{-1}-(T_\rho+V)^{-1})$ exists (see, for example, (5)). Calculating

Expanding $\mathrm{Sp}\{T_\rho^{-1}-(T_\rho+V)^{-1}\}$ in the basis of eigenfunctions ψ_n of the perturbed operator $T+V$, and then in the basis of eigenfunctions of the unperturbed operator T, we obtain the equality

$$\sum_{n=1}^{\infty}\left\{\frac{1}{\lambda_n+\rho}-\left((T_\rho+V)^{-1}\varphi_n,\varphi_n\right)\right\}=\sum_{n=1}^{\infty}\left\{\left(T_\rho^{-1}\psi_n,\psi_n\right)-\frac{1}{\mu_n+\rho}\right\}. \quad (14)$$

Consider the partial sums of these series and use the result of Ky Fan [6], according to which

$$\sum_{n=1}^{N}\left((T_\rho+V)^{-1}\varphi_n,\varphi_n\right)\le\sum_{n=1}^{N}\frac{1}{\mu_n+\rho}$$

and

$$\sum_{n=1}^{N}\left(T_\rho^{-1}\psi_n,\psi_n\right)\le\sum_{n=1}^{N}\frac{1}{\lambda_n+\rho}.$$

We obtain

$$\sum_{n=1}^{N}\left\{\left(T_\rho^{-1}\psi_n,\psi_n\right)-\frac{1}{\mu_n+\rho}\right\}\le\sum_{n=1}^{N}\left(\frac{1}{\lambda_n+\rho}-\frac{1}{\mu_n+\rho}\right)\le\sum_{n=1}^{N}\left\{\frac{1}{\lambda_n+\rho}-\left((T_\rho+V)^{-1}\varphi_n,\varphi_n\right)\right\},$$

which, as $N\to\infty$, by virtue of equality (14), gives (13). From (13) it is already not difficult to obtain the equality

$$\lim_{\rho\to\infty}\rho^2\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n+\rho}-\frac{1}{\mu_n+\rho}\right)=\lim_{\rho\to\infty}\rho^2\{\mathrm{Sp}(T_\rho^{-1}VT_\rho^{-1})-\mathrm{Sp}(T_\rho^{-1}(VT_\rho^{-1})^2)\}. \quad (15)$$

It is easy to show further that if the series $\sum_{n=1}^{\infty}(\mu_n-\lambda_n)$ converges, then the series

$$\rho^2\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n+\rho}-\frac{1}{\mu_n+\rho}\right)=\rho^2\sum_{n=1}^{\infty}\frac{\mu_n-\lambda_n}{(\lambda_n+\rho)(\mu_n+\rho)} \quad (16)$$

converges uniformly with respect to ρ. The proof of this fact, given in [2], relies only on the monotonicity of λ_n and μ_n, which we also assume, since we consider them arranged in increasing order. The termwise passage to the limit $\rho \to \infty$ in (16) completes the proof of the theorem.

In conclusion, the author takes the opportunity to express sincere gratitude to Prof. V. B. Lidskii for the formulation of the problem and for his constant attention to the work.

Moscow Institute of Physics and Technology

Received
5 V 1962

References

[1] W. A. Ambartsumjian, Zs. f. Phys., **53**, 690 (1929).
[2] C. Halberg, V. Kramer, Duke Math. J., **27**, No. 4 (1960).
[3] Ch. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Differential Equations*, 2, Moscow, 1961.
[4] R. Courant, D. Hilbert, *Methods of Mathematical Physics*, 1, Moscow, 1951.
[5] V. B. Lidskii, Tr. Moskovsk. matem. obshch., **8**, 83 (1959).
[6] Ky Fan, Proc. Nat. Acad. Sci. USA, **35**, No. 9 (1949).

Note: Figure translations are in progress. See original paper for figures.