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Abstract

Full Text

MATHEMATICS

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A WIENER-TYPE THEOREM FOR SERIES IN WALSH FUNCTIONS

(Presented by Academician V. I. Smirnov on 22 IX 1961)

In the present note it will be shown that, for functions expandable in an absolutely convergent series in Walsh functions, Wiener-type theorems remain valid.

As is known ⁽²⁾, the Rademacher functions are defined as follows:

$$\varphi_0(x) = 1 \quad (0 \leq x < 1/2); \quad \varphi_0(x) = -1 \quad (1/2 \leq x < 1);$$

$$\varphi_0(x+1) = \varphi_0(x); \quad \varphi_n(x) = \varphi_0(2^n x), \quad n = 1, 2, \dots,$$

and the Walsh functions, forming an orthonormal system, are defined by the relations:

$$\psi_0(x) = 1, \quad \psi_n(x) = \varphi_{n_1}(x)\varphi_{n_2}(x) \dots \varphi_{n_r}(x), \quad 0 \leq x \leq 1,$$

for $n = 2^{n_1} + 2^{n_2} + \dots + 2^{n_r}$, where the nonnegative integers n_i are uniquely determined by the number n and by the inequality $n_{i+1} < n_i$.

Consider the collection W of functions expandable in an absolutely convergent series in Walsh functions on the interval $[0, 1]$:

$$f(x) = \sum_{n=0}^{\infty} a_n \psi_n(x), \quad \sum_{n=0}^{\infty} |a_n| < \infty.$$

It is easy to verify that the indicated collection of functions forms a linear space.

Theorem 1. Let $f(x) \in W$ and $\inf |f(x)| > 0$.

Then $\frac{1}{f(x)}$ is expandable in an absolutely convergent series

$$\frac{1}{f(x)} = \sum_{n=0}^{\infty} b_n \psi_n(x), \quad \sum_{n=0}^{\infty} |b_n| < \infty.$$

The proof of the theorem will be given below.

Let us note that the functions $f(x) \in W$ are continuous at dyadic irrational points and may have discontinuities of the first kind at dyadic rational points. Therefore we shall pass from the interval $[0, 1]$ to a somewhat more complicated set A , which is obtained if each dyadic rational point x is replaced by a pair of points $x-0, x+0$. We introduce a topology on the set A by means of the system of neighborhoods of the form $[a+0, b-0]$, where a and b are dyadic rational points, $a < b$. In this topology the set A is a bicomact zero-dimensional space, homeomorphic to the Cantor perfect set. On the space A all functions $f \in M$ become single-valued and continuous.

We introduce a norm of elements of W , setting

$$\|f\| = \sum_{n=1}^{\infty} |a_n|.$$

It is easy to verify that the norm axioms are satisfied.

In view of the completeness of the l_1 -space of absolutely convergent numerical series, the isometric space W will also be a complete normed space.

Further, if $f(x) \in W, g(x) \in W$, then $f(x)g(x) \in W$. Indeed, let

$$f(x) = \sum_{n=0}^{\infty} a_n \psi_n(x), \quad g(x) = \sum_{m=0}^{\infty} b_m \psi_m(x).$$

Then

$$f(x)g(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m \psi_n(x) \psi_m(x).$$

We shall prove that

$$f(x)g(x) = \sum_{k=0}^{\infty} c_k \psi_k(x), \quad \sum_{k=0}^{\infty} |c_k| < \infty.$$

It is easy to verify that the product of any two Walsh functions belongs to the Walsh system. Indeed, taking into account that the product of two Rademacher functions with one and the same index is identically equal to one, one may assert that for any Walsh functions

$$\psi_{n_0}(x) = \varphi_{n_1}(x) \varphi_{n_2}(x) \dots \varphi_{n_r}(x),$$

$$\psi_{m_0}(x) = \varphi_{m_1}(x)\varphi_{m_2}(x) \dots \varphi_{m_p}(x),$$

we shall have:

$$\varphi_{n_0}(x)\psi_{m_0}(x) = \varphi_{k_1^0}(x)\varphi_{k_2^0}(x) \dots \varphi_{k_n^0}(x) = \psi_{k^0}(x),$$

where

$$k^0 = 2^{k_1^0} + 2^{k_2^0} + \dots + 2^{k_n^0}.$$

Consequently,

$$f(x)g(x) = \sum_{k=0}^{\infty} c_k \psi_k(x).$$

Moreover, since

$$\sum_{k=1}^{\infty} |c_k| \leq \sum_{n=0}^{\infty} |a_n| \sum_{n=0}^{\infty} |b_n| < \infty,$$

then

$$h(x) \equiv f(x)g(x) \in W.$$

On the other hand, the inequality

$$\|h\| = \sum_{k=0}^{\infty} |c_k| \leq \sum_{n=0}^{\infty} |a_n| \cdot \sum_{m=0}^{\infty} |b_m| = \|f\| \cdot \|g\|$$

shows that multiplication is continuous in the norm introduced in W . Thus, W is a normed ring. In order that $f \in W$ be regular, it is necessary and sufficient that, for every multiplicative functional F , one have $F(f) \neq 0$. Let us find the general form of the nontrivial multiplicative functionals in W .

For any $f \in W$ we have $|F(f)| \leq \|f\|$, and also $F(1) = 1$. We shall prove that every multiplicative functional in W is represented by the value of the function $f(x)$ at some point ξ of the space A .

We first prove the following important lemma:

Lemma. Let ε_k ($k = 0, 1, \dots$) be an arbitrary sequence of numbers equal to $+1$ or -1 . Then in the space A there exists a unique point ξ such that, for the Rademacher functions, the equalities $\varphi_k(\xi) = \varepsilon_k$ hold.

Proof. Let $\varphi_1(x) = \varepsilon_1$ on the interval Δ_1 (Δ_1 is either $[0, \frac{1}{2} - 0]$, or $[\frac{1}{2} + 0, 1]$). Recall that we distinguish the points $\frac{1}{2} - 0$ and $\frac{1}{2} + 0$. Denote by Δ_2 that part of Δ_1 where $\varphi_2(x) = \varepsilon_2$. Continuing this process, we obtain a sequence $\{\Delta_k\}$, where $\text{mes } \Delta_k = 1/2^k \rightarrow 0$ as $k \rightarrow \infty$. By virtue of the bicomactness of the space A , in the intersection of the intervals Δ_k there is a unique point $\xi \in A$, for which $\varphi_k(\xi) = \varepsilon_k$, $k = 1, 2, \dots$

Theorem 2. For any prescribed multiplicative functional F in W , there exists a point $\xi \in A$ such that, for every function $f(x) \in W$,

$$F[f] = f(\xi).$$

Proof. Since the generators of the ring W are the Rademacher functions, it is enough to prove (1) for the case when $f(x)$ is a Rademacher function.

In view of the fact that $\|\varphi_k\| = 1$, we have $|F(\varphi_k)| \leq \|\varphi_k\| = 1$. Since $\varphi_k^2(x) \equiv 1$, it follows that $F^2[\varphi_k] = 1$. Consequently, $F(\varphi_k) = \varepsilon_k$. But, according to the lemma, there exists, and moreover a unique, point ξ such that $\varphi_k(\xi) = \varepsilon_k$, i.e. $F[\varphi_k] = \varphi_k(\xi)$. This relation holds on the generators, and hence on every element of the ring, i.e. formula (1) holds.

Proof of the main Theorem 1. Let F be an arbitrary multiplicative functional. By Theorem 2 there exists a point ξ , depending on the choice of F , such that formula (1) holds. By hypothesis $\inf|f| > 0$, and consequently $F(f) \neq 0$. Since this holds for every multiplicative functional F , it may be asserted, according to what was said above, that f is a regular element of W , i.e.

$$\frac{1}{f(x)} = \sum_{n=0}^{\infty} b_n \varphi_n(x), \quad \sum_{n=0}^{\infty} |b_n| < \infty.$$

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Note: Figure translations are in progress. See original paper for figures.

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