MATHEMATICS

Abstract

Full Text

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INFINITESIMAL BENDINGS OF A CONVEX SURFACE WITH A BOUNDARY CONDITION OF GENERALIZED SLIDING

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1. Let, with respect to the coordinate system $Oxyz$, a simply connected convex surface S with positive Gaussian curvature $K \geq k_0 > 0$ be given by the equation $z = f(x, y)$; let Γ be the boundary of S. For the derivatives of the function $f(x, y)$ we shall use Monge notation: $p = \partial f/\partial x$, $q = \partial f/\partial y$, $r = \partial^2 f/\partial x^2$, $s = \partial^2 f/\partial x \partial y$, $t = \partial^2 f/\partial y^2$. Let $\mathbf{U}(x, y) = \alpha(x, y)\mathbf{i} + \beta(x, y)\mathbf{j} + \gamma(x, y)\mathbf{k}$ be the vector of an infinitesimal bending of the surface S; G the domain in the plane Oxy onto which S is projected; L the boundary of the domain G. For brevity, in what follows, instead of "infinitesimal bending" we shall simply write "bending."

It is known ([1,2]) that, under the assumptions made, the surface S admits no nontrivial bendings for which the vector $\mathbf{U}(x, y)$ is perpendicular along Γ to one and the same vector $\mathbf{k}$. (Such bendings may be called bendings of sliding parallel to the plane Oxy.) In this case there exist exactly three linearly independent trivial bendings ([2]). But if one additionally requires that the displacement vector be equal to zero at some point $M \in \Gamma$, and that at this point one nonparallel-to-the-plane Oxy direction l be forbidden for the rotation vector, then the surface S will not even admit motions in space as a rigid body. (Saying that a certain direction l at the point M is forbidden for the rotation vector $\mathbf{V}$, we mean that $\mathbf{V}$ is perpendicular to l at M.) In the present work this fact is generalized to the case when the vector $\mathbf{U}$ is perpendicular along Γ to a prescribed variable vector field there. Let a field of vectors $(a, b, -1)$ be prescribed along Γ, where a and b are functions defined on L and satisfying the condition $1 + ap + bq \neq 0$ on L. (The latter means that the given field is nowhere tangent to S.) Considering the prescribed field as a field of normals along Γ to some surface Ω, the principal result of the work may be formulated as follows: *if the surface Ω differs sufficiently little along Γ from the plane Oxy in the sense of smallness (in some norm) of the divergence of their normals, then S admits exactly three linearly independent bendings* (in general, nontrivial ones), *for which $\mathbf{U}$ is perpendicular along Γ to the field prescribed there.* These bendings, by analogy with what was said above, may be called bendings of sliding along Ω. In this case, if one fixes some point $M \in \Gamma \in S$ and forbids at this point, for the vector $\mathbf{V}$, the direction of the normal $\mathbf{n}$ to S, then S will admit no bendings. At the end of the work an example is given showing that the condition of smallness of the divergence of the vectors $-\mathbf{k}$ and $(a, b, -1)$ is essential, i.e., that otherwise bendings of sliding along Ω are possible.

2. The condition of perpendicularity of the vector $\mathbf{U}$ to the field prescribed along Γ leads to a linear relation on the boundary between the components of $\mathbf{U}$:

$$\gamma = a\alpha + b\beta \quad \text{on } L. \tag{1}$$

The case $1 + ap + bq \equiv 0$ is considered in detail in ([3]); there the general problem of the form (1) is also formulated. Its special cases (when the angle between the binormal Γ and the given field is constant) were studied in ([4]), where a lower bound is estimated for the number of possible linearly independent bendings. Other geometric problems also lead to a condition of the form (1), for example, the problem of a bending under which the distance from Γ to some fixed point remains stationary.

3. Suppose that $f(x, y) \in C^2_\sigma(G + L)$, $L \in C^2_\sigma$, and a and $b \in C^2_\sigma(L)$ (for the definition of these classes see ([3])). As A. V. Pogorelov showed in ([2]), under these conditions the bending field belongs to the class $C^2_\sigma(G + L)$, $0 < \sigma < \sigma' \leqslant 1$.

The set of functions φ of class $C^2_\sigma(L)$ becomes a Banach space with the norm $\|\varphi\|_{2,\sigma}(L)$, introduced as in ([3]). We formulate the exact result of the work:

Theorem 1. *Let a vector field $(a, b, -1)$ be prescribed along Γ, with the condition $1 + ap + bq \neq 0$. Then there exist numbers $Q > 0$, $P > 0$, $Q < P$, depending only on the surface S and the domain G, such that:*

1) if

$$\left\| \frac{a + ib}{1 + ap + bq} \right\|_{2,\sigma} Q \leqslant q_0 < 1,$$

then the surface S admits exactly three linearly independent bendings satisfying condition (1);

2) if

$$\left\| \frac{a + ib}{1 + ap + bq} \right\|_{2,\sigma} P \leqslant p_0 < 1,$$

then the surface S admits no bendings (including trivial ones) for which, in addition to condition (1), the following conditions are also satisfied: $\mathbf{U}(M) = 0$, $\mathbf{V}(M) \perp \mathbf{n}(M)$; $\mathbf{n}$ is the normal to S; M is some point on Γ.

The proof is carried out as follows: the components $\alpha(x, y)$ and $\beta(x, y)$ of the vector $\mathbf{U}(x, y)$ are expressed in terms of $\gamma(x, y)$, proceeding from the equations of bending of the surface S.

The component $\gamma(x,y)$, in turn, satisfies the elliptic equation

$$f_{yy}\gamma_{xx} - 2f_{xy}\gamma_{xy} + f_{xx}\gamma_{yy} = 0. \tag{2}$$

We take an arbitrary function $\gamma(t) \in C^2_\sigma(L)$ and solve the corresponding Dirichlet problem for equation (2). Then the functions $\alpha(x,y)$, $\beta(x,y)$, and $\gamma(x,y)$ will be expressed through $\gamma(t)$ with the aid of certain operators. Substituting their boundary values into the boundary condition (1), we obtain a homogeneous linear functional equation on L with respect to $\gamma(t)$, whose uniqueness (zero) solution is ensured under the conditions of the theorem.

The most essential point in these arguments is obtaining a formula for expressing $\alpha(x,y)$ and $\beta(x,y)$ only through $\gamma(x,y)$ (without the participation of its derivatives).

We introduce the functions $\lambda = \alpha + p\gamma$, $\mu = \beta + q\gamma$, $W = \lambda + i\mu$, $z = x + iy$. For them the equations of infinitesimal bendings of the surface may be written as

$$\frac{\partial W}{\partial \bar z} = 2\,\frac{\partial^2 f}{\partial z^2}\,\gamma(z), \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{3$_1$}$$

$$2\ \mathrm{Re}\left(\frac{\partial W}{\partial \bar z}\right) = 4\,\frac{\partial^2 f}{\partial z\,\partial \bar z}\,\gamma(z), \qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{3$_2$}$$

We assume that L is the circle $x^2 + y^2 = 1$. The general case is reduced to this by a conformal transformation. According to the results of ([3]), the general solution of equation (3_1) is represented in the form

$$W(z) = \frac{1}{2\pi i}\int_L \frac{W(\tau)}{\tau - z}\,d\tau - \frac{2}{\pi}\iint_G \frac{\dfrac{\partial^2 f}{\partial \bar\zeta^2}\,\gamma(\zeta)}{\zeta - z}\,d\xi\,d\eta, \qquad \zeta = \xi + i\eta. \tag{4}$$

From relation (4) and equation (3_2) we can find $\partial W/\partial z$, using the Schwarz formula for determining an analytic function in a disk from the values of its real part on the circumference. Taking into account that

$$\frac{dW}{dt} = \frac{\partial W}{\partial t} + \frac{\partial W}{\partial \bar t}\,{t'}^2(s),$$

we obtain

$$W(t) = W_1(t) + iv_0(t-\tau_0) + v_1 + iv_2 =$$
$$= \int_{\tau_0}^{t} \left\{ \frac{d}{d\tau}\left(\frac{\partial f}{\partial \overline{\tau}}\right) + \operatorname{Re}\left[\frac{d}{d\tau}\left(\frac{\partial f}{\partial \overline{\tau}}\right)\right]\gamma(\tau)\right\} d\tau - \frac{2i}{\pi}\int_{\tau_0}^{t} d\tau \operatorname{Im} \iint_G \frac{\frac{\partial^2 f}{\partial \overline{\zeta}^2}\gamma(\zeta)}{(\overline{\zeta}-\tau)^2}\, d\xi\, d\eta$$
$$+ \int_{\tau_0}^{t} d\tau \frac{1}{\pi i}\int_L \frac{\operatorname{Re}\left\{ \left[\frac{d}{dt_1}\left(\frac{\partial f}{\partial \overline{t}_1}\right) + \frac{\partial^2 f}{\partial t_1 \partial \overline{t}_1}\right]\gamma(t_1) + \frac{2}{\pi}\iint_G \frac{\frac{\partial^2 f}{\partial \overline{\zeta}^2}\gamma(\zeta)}{(\overline{\zeta}-t_1)^2}\, d\xi\, d\eta \right\}}{t_1 - \tau}\, dt_1$$
$$+ u_0(t-\tau_0) + iv_0(t-\tau_0) + v_1 + iv_2, \tag{5}$$

where

$$u_0 = -2\int_0^{2\pi} \operatorname{Re}\left\{ \left[\frac{d}{dt}\left(\frac{\partial f}{\partial \overline{t}}\right) + \frac{\partial^2 f}{\partial t\, \partial \overline{t}}\right]\gamma(t(s)) + \frac{2}{\pi}\iint_G \frac{\frac{\partial^2 f}{\partial \overline{\zeta}^2}\gamma(\zeta)}{(\overline{\zeta}-t)^2}\, d\xi\, d\eta \right\} ds;$$

v_0, v_1, and v_2 are arbitrary real constants; τ_0 is an arbitrary fixed point on L. Let us now estimate $\|W_1(t)\|_{2,\sigma}$. For this it is necessary to take into account that for solutions of the Dirichlet problem for equation (2) formula (5) implies

$$\|\gamma\|_{2,\sigma}(G+L) = O(\|\gamma\|_{2,\sigma}(L)). \tag{6}$$

Applying now the corresponding estimates for the norms of Cauchy-type integrals and of the double integral

$$\iint_G \frac{\frac{\partial^2 f}{\partial \overline{\zeta}^2}\gamma(\zeta)}{(\overline{\zeta}-\tau)^2}\, d\xi\, d\eta$$

from (3), we obtain that

$$\|W_1(t)\|_{2,\sigma} \le Q\|\gamma(t)\|_{2,\sigma}, \tag{7}$$

where Q depends on $f(x,y)$ and G.

The boundary condition (1) for the functions λ, μ, and γ takes the form

$$\gamma(t) = \operatorname{Re}\left[\frac{a+ib}{1+ap+bq}\,W(t)\right] =$$

$$= \operatorname{Re}\left[\frac{a+ib}{1+ap+bq}\,W_1(t)\right] + \operatorname{Re}\left[\frac{a+ib}{1+ap+bq}\,(iv_0(t-\tau_0)+v_1+iv_2)\right], \quad (8)$$

whence, with estimate (7) taken into account, the first assertion of the theorem is easily obtained by applying the method of successive approximations.

To prove its second part, it is enough to observe that we can choose as τ_0 the point on L corresponding to the point $M \in \Gamma$. Then

$$v_1 = v_2 = 0,$$

$$v_0 = -\operatorname{Im}\frac{1}{\pi i}\int_L \frac{\operatorname{Re}\left\{\left[\frac{d}{dt}\left(\frac{\partial f}{\partial t}\right) + \frac{\partial^2 f}{\partial t\,\partial \bar{t}}\right]\gamma(t) + \frac{2}{\pi}\iint_G \frac{\frac{\partial^2 f}{\partial \bar{\zeta}^2}\gamma(\zeta)}{(\bar{\zeta}-t)^2}\,d\xi\,d\eta\right\}}{t-\tau_0}\,dt +$$

$$+\operatorname{Im}\frac{2}{\pi}\iint_G \frac{\frac{\partial^2 f}{\partial \bar{\zeta}^2}\gamma(\zeta)}{(\bar{\zeta}-\tau_0)^2}\,d\xi\,d\eta.$$

Obviously,

$$|v_0| \leqslant K_1\|\gamma\|_C(L) + K_2\|\gamma\|_{0,\sigma}(G+L),$$

where K_1 and K_2 depend on $f(x,y)$ and L. Hence, from (6) we obtain that

$$\|W(t)\|_{2,\sigma}(L) \leqslant P\|\gamma(t)\|_{2,\sigma}(L). \quad (9)$$

Having the estimate (9), we again prove the second part of the theorem by the method of successive approximations.

4. Let us give an example which shows that the smallness condition given in the theorem is essential. Put $f(x,y) = x^2+y^2$; let G be the disk

$$x^2+y^2 \leqslant \frac{k+1}{k-1} = R^2, \qquad k \geqslant 2$$

where k is an integer, and

$$a(s) = -\frac{k-1}{2k} R\cos s, \qquad b(s) = -\frac{k-1}{2k} R\sin s, \qquad 0 \leqslant s \leqslant 2\pi.$$

We take as the point M the point $(R, 0, R^2)$. It is not hard to verify that the functions

$$\begin{aligned}
\alpha(x,y) = &-4\ \mathrm{Im}\, z^{k+1} + 4(k+1)x\ \mathrm{Im}\, z^k - 4(k^2-1)R^k y,\\
\beta(x,y) = &\ 4\ \mathrm{Re}\, z^{k+1} + 4(k+1)y\ \mathrm{Im}\, z^k - 2k(k-1)R^{k+1}(x^2+y^2)\\
&+ 4(k^2-1)R^k x - 2k(k-1)R^{k+1},\\
\gamma(x,y) = &-2(k+1)\,\mathrm{Im}\, z^k + 2k(k-1)R^{k+1} y
\end{aligned}$$

define bendings of the surface, vanish together with their derivatives with respect to x and y at M, and satisfy the boundary condition

$$\gamma = -\frac{(k-1)x}{2k}\,\alpha - \frac{(k-1)y}{2k}\,\beta \quad \text{on } L:\ x^2+y^2 = \frac{k+1}{k-1}.$$

5. From formula (5) one can derive one consequence. Suppose we are given two convex surfaces S_1 and S_2 of positive curvature with a common edge, both projected one-to-one onto the plane Oxy, and both turned with their convexities to the same side (this gluing may be called internal, since one surface enters inside the other). The vector U changes continuously on the surface $S = S_1 + S_2$. Again expressing $\alpha(x,y)$ and $\beta(x,y)$ on the edge Γ in terms of $\gamma(t)$ by formula (5), from the condition of continuity of α and β in passing through Γ we obtain the functional equation for $\gamma(t)$ on L

$$[(p_1-p_2) + i(q_1-q_2)]\gamma(t) = A_1(\gamma) - A_2(\gamma),$$

where A_i is an operator of the form (5) for S_1 and S_2. Hence follows

Theorem 2. If the norm

$$\left\| \frac{1}{p_1 - p_2 + i(q_1-q_2)} \right\|_{2,\sigma}$$

is sufficiently small, then the closed surface $S = S_1 + S_2$ is rigid.

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Note: Figure translations are in progress. See original paper for figures.