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Abstract

Full Text

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ON HALL' S THEOREM

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§ 1

The main result of the present note is Theorem 2, which generalizes theorem D5 of Ph. Hall from his paper ([1]). The notation used is taken from ([1]).

Let Π be some (empty or nonempty) set of primes; $\mathfrak{G}$ a finite group of order $o(\mathfrak{G}) = g = mn$, where $m \geq 1$ is the greatest Π-divisor ([2]) of the order g; a subgroup $\mathfrak{H}$ of order m of the group $\mathfrak{G}$ will be called an S_Π-subgroup of the group $\mathfrak{G}$.

We shall consider the following properties of finite groups: E_Π—$\mathfrak{G}$ has at least one S_Π-subgroup; E_Π^n—$\mathfrak{G}$ has a nilpotent S_Π-subgroup; C_Π—$\mathfrak{G}$ has property E_Π, and any two S_Π-subgroups of the group $\mathfrak{G}$ are conjugate in $\mathfrak{G}$; D_Π—$\mathfrak{G}$ has property C_Π, and every Π-subgroup of the group $\mathfrak{G}$ is contained in some S_Π-subgroup of the group $\mathfrak{G}$; D_Π^s—$\mathfrak{G}$ has property D_Π, and its S_Π-subgroups are solvable.

We now formulate the aforementioned theorem of Ph. Hall.

Theorem 1 (([1]), theorem D5). *If $\mathfrak{K}$ is such a normal divisor of the group $\mathfrak{G}$ that $\mathfrak{K}$ has property E_Π^n, and $\mathfrak{G}/\mathfrak{K}$ has property D_Π^s, then $\mathfrak{G}$ has property D_Π^s.*

§ 2

Definition. A Π-subgroup Ω of the group $\mathfrak{G}$ will be called a Π-suitable subgroup of the group $\mathfrak{G}$ if $\mathfrak{G}$ has property E_Π and in each conjugacy class of S_Π-subgroups of the group $\mathfrak{G}$ there is at least one S_Π-subgroup containing Ω.

From this definition the following proposition follows:

A. *A finite group has property D_Π if and only if all its Π-subgroups are Π-suitable.*

Theorem 2. *Let $\mathfrak{G}$ have an invariant subgroup $\mathfrak{K}$ possessing property E_Π^n, and let Ω be such a Π-subgroup of the group $\mathfrak{G}$ that $\Omega\mathfrak{K}/\mathfrak{K}$ is a solvable Π-suitable subgroup of the group $\mathfrak{G}/\mathfrak{K}$. Then Ω is a solvable Π-suitable subgroup of the group $\mathfrak{G}$.*

Proof. Suppose that the theorem is false. Then choose, among the groups for which the theorem is not satisfied, a group $\mathfrak{G}$ of least order. Thus, $\mathfrak{G}$ has an invariant subgroup $\mathfrak{K}$ possessing property E_{Π}^{n}, and in $\mathfrak{G}$ there exists such a Π-subgroup Ω that $\Omega\mathfrak{K}/\mathfrak{K}$ is a solvable Π-suitable subgroup of the group $\mathfrak{G}/\mathfrak{K}$, but Ω is not a solvable Π-suitable subgroup of the group $\mathfrak{G}$.

By the hypothesis of the theorem and the definition, $\mathfrak{G}/\mathfrak{K}$ has property E_{Π}. By G. Wielandt' s theorem ([3]), $\mathfrak{K}$ has property C_{Π}. Hence $\mathfrak{G}$ has property E_{Π} by theorem E2 from ([1]).

Let $\mathfrak{H}$ be an arbitrary S_{Π}-subgroup of the group $\mathfrak{G}$. We have to prove that Ω is solvable and $\Omega \subseteq \mathfrak{H}^{G}$, where $G \in \mathfrak{G}$. By lemma 1 from ([1]), $\mathfrak{H}\mathfrak{K}/\mathfrak{K}$ is an S_{Π}-subgroup of the group $\mathfrak{G}/\mathfrak{K}$. Since, by hypothesis, $\Omega\mathfrak{K}/\mathfrak{K}$ is a Π-suitable subgroup of the group $\mathfrak{G}/\mathfrak{K}$, it follows that

$$\Omega\mathfrak{K}/\mathfrak{K} \subseteq (\mathfrak{H}\mathfrak{K}/\mathfrak{K})^{G_1} = \mathfrak{H}^{G_1}\mathfrak{K}/\mathfrak{K},$$

where $G_1 \in \mathfrak{G}$. Hence it follows that $\Omega \subseteq \mathfrak{H}^{G_1}\mathfrak{K}$. Consider two cases:

1. $(\mathfrak{H}^{G_1}\mathfrak{K}) < (\mathfrak{G})$. Since $\mathfrak{H}^{G_1}\mathfrak{K}/\mathfrak{K}$ is a Π-subgroup, $\mathfrak{L}\mathfrak{K}/\mathfrak{K}$ is a solvable Π-suitable subgroup of the group $\mathfrak{H}^{G_1}\mathfrak{K}/\mathfrak{K}$. Hence, by the induction hypothesis, it follows that $\mathfrak{L}$ is solvable and $\mathfrak{L} \subseteq \mathfrak{H}^{G_1 G_2}$, where $G_2 \in \mathfrak{H}^{G_1}\mathfrak{K}$. We have arrived at a contradiction.
2. $\mathfrak{H}^{G_1}\mathfrak{K} = \mathfrak{G}$. Then also $\mathfrak{H}^{G_1}\mathfrak{K}\mathfrak{L} = \mathfrak{G}$. Let $\mathfrak{H}_1 = \mathfrak{H}^{G_1} \cap \mathfrak{K}\mathfrak{L}$. Obviously, the relation
$$(\mathfrak{G}) = (\mathfrak{H}^{G_1} \cdot \mathfrak{K}\mathfrak{L}) = \frac{(\mathfrak{H}^{G_1})(\mathfrak{K}\mathfrak{L})}{(\mathfrak{H}_1)}$$
holds.

It follows from this that $(\mathfrak{G} : \mathfrak{H}^{G_1}) = (\mathfrak{K}\mathfrak{L} : \mathfrak{H}_1)$. Consequently, $\mathfrak{H}_1$ is an S_{Π}-subgroup of the group $\mathfrak{K}\mathfrak{L}$. By hypothesis, $\mathfrak{K}\mathfrak{L}/\mathfrak{K}$ is solvable, and $\mathfrak{K}$ has property E_{Π}^{n}. Hence, by Theorem 1, $\mathfrak{K}\mathfrak{L}$ has property D_{Π}^{s}. Therefore $\mathfrak{L}$ is solvable and $\mathfrak{L} \subseteq \mathfrak{H}_1^{G_2}$, where $G_2 \in \mathfrak{K}\mathfrak{L}$. Since $\mathfrak{H}_1 \subseteq \mathfrak{H}^{G_1}$, we have

$$\mathfrak{L} \supseteq \mathfrak{H}_1^{G_2} \subseteq \mathfrak{H}^{G_1 G_2},$$

where $G_1 G_2 \in \mathfrak{G}$. Again we have arrived at a contradiction.

Thus, the supposition of the existence of groups for which the theorem is false always leads to a contradiction. The theorem is thereby proved.

It is easy to see that Theorem 1 follows from Theorem 2, taking into account assumption A.

Theorem 3. *Let $\mathfrak{G}$ have an invariant subgroup $\mathfrak{K}$ possessing property E_{Π}^{n}, and let $\mathfrak{L}$ be such a Π-subgroup of the group $\mathfrak{G}$ that $\mathfrak{L}\mathfrak{K}/\mathfrak{K}$ is solvable and is contained in at least one S_{Π}-subgroup of the group $\mathfrak{G}/\mathfrak{K}$. Then $\mathfrak{L}$ is solvable and is contained in at least one S_{Π}-subgroup of the group $\mathfrak{G}$.*

Proof. According to the hypothesis of the theorem,

$$\mathfrak{L}\mathfrak{K}/\mathfrak{K} \subseteq \mathfrak{H}^{*}/\mathfrak{K},$$

where $\mathfrak{H}^*/\mathfrak{K}$ is some S_Π-subgroup of the group $\mathfrak{G}/\mathfrak{K}$. Hence, $\mathfrak{L} \subseteq \mathfrak{H}^*$. Since $\mathfrak{H}^*/\mathfrak{K}$ is a Π-group, $\mathfrak{L}\mathfrak{K}/\mathfrak{K}$ is a solvable Π-suitable subgroup of the group $\mathfrak{H}^*/\mathfrak{K}$. Then, by Theorem 2, $\mathfrak{L}$ is a solvable Π-suitable subgroup of the group $\mathfrak{H}^*$. Thus, $\mathfrak{L} \subseteq \mathfrak{H}$, where $\mathfrak{H}$ is some S_Π-subgroup of $\mathfrak{H}^*$. It is easy to see that $(\mathfrak{H}) = m$, i.e. $\mathfrak{H}$ is an S_Π-subgroup of the group $\mathfrak{G}$. The theorem is thereby proved.

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[3] H. Wielandt, Math. Zs., 60, 407 (1954).

Note: Figure translations are in progress. See original paper for figures.