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Abstract

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MATHEMATICS

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ON CRITERIA FOR COMPLETE CONTINUITY OF LINEAR AND NONLINEAR INTEGRAL OPERATORS

(Presented by Academician S. L. Sobolev on 1 VIII 1961)

The aim of the present article is to establish certain properties of integral operators in the spaces $L_p = L_p(\Omega)$ $(\operatorname{mes}\Omega < \infty,\ p \geqslant 1)$.

1. Below we use the following simple compactness criterion in the spaces L_p $(1 \leqslant p < \infty)$ for the case of Orlicz spaces, proved in [1].

Lemma 1. *A family of functions in L_p is compact if and only if it is compact in measure and if the p-th powers of the functions have equiabsolutely continuous integrals.*

An operator defined on some L_p will be called **compact in measure** if it maps every bounded set in L_p into a family of functions compact in measure. For example, every completely continuous operator acting from L_p into L_r is compact in measure.

Consider the linear integral operator

$$Au(t) = \int_\Omega K(t,s)u(s)\,ds \tag{1}$$

with a nonnegative kernel $K(t,s)$, measurable in the joint variables $(t \in \Omega^*,\ s \in \Omega)$. If the kernel $K(t,s)$ is essentially bounded, then A is a completely continuous operator acting from any L_p into any L_r $(r < \infty)$ [2]. We note that the operator A need not possess the property of complete continuity as an operator from L_1 to L_∞ for a bounded kernel $K(t,s)$ (a simple example was pointed out to us by A. I. Perov). We shall be interested in cases of unbounded kernels $K(t,s)$.

Lemma 2. *The integral operator (1), bounded as an operator from $L_p(\Omega)$ to $L_r(\Omega^*)$ $(1 \leqslant p, r < \infty)$, is compact in measure (cf. [2]).*

For the proof it is enough to show that A is completely continuous as an operator from L_p to L_1. Pass to the adjoint operator

$$A^*u(s) = \int_{\Omega^*} K(t,s)u(t)\,dt. \tag{2}$$

On the basis of general theorems of functional analysis, it is a bounded operator acting from $L_{r'}(\Omega^*)$ to $L_{p'}(\Omega)$, where

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad \frac{1}{r} + \frac{1}{r'} = 1.$$

The function $u_0(t) \equiv 1$ belongs to $L_{r'}(\Omega^*)$; hence $\varphi_0(s) = A^* u_0 \in L_{p'}(\Omega)$. We now estimate the norm of $Au(t)$ in $L_1(\Omega^*)$:

$$\|Au(t)\|_{L'} \leqslant \int_{\Omega^*} \int_{\Omega} K(t,s)|u(s)|\,ds\,dt \leqslant \|\varphi_0(s)\|_{L_{p'}} \|u(t)\|_{L_p}. \tag{3}$$

Introduce the notation

$$K_n(t,s) = \min\{K(t,s);\, n\} \qquad (n = 1, 2, \dots) \tag{4}$$

and set

$$A_n u(t) = \int_{\Omega} K_n(t,s)\, u(s)\, ds. \tag{5}$$

All the operators A_n act completely continuously from L_p into L_1. On the basis of inequality (3),

$$\|(A - A_n)u(t)\|_{L'} \leq \|\varphi_n(s)\|_{L_{p'}} \|u(t)\|_{L_p},$$

where

$$\varphi_n(s) = \int_{\Omega^*} |K(t,s) - K_n(t,s)|\, dt.$$

Let $\varphi_0(s_0) < \infty$, where s_0 is a fixed point. Then $\varphi_n(s_0) \to 0$ as $n \to \infty$. But $\varphi_0(s)$ is finite almost everywhere. Thus the sequence $\varphi_n(s)$ converges to zero in measure and is bounded by the function $\varphi_0(s)$ from $L_{p'}$. From Lebesgue' s theorem on passage to the limit under the integral sign (3) it follows that

$$\lim_{n\to\infty} \|\varphi_n(s)\|_{L_{p'}} = 0.$$

We have shown that the operator A is the uniform limit of completely continuous operators A_n and, consequently, is itself completely continuous. The lemma is proved.

2. Theorem 1. *Let the operator* (1) *be bounded as an operator from* L_{p_0} *to* L_{r_0}.

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Then it is a completely continuous operator acting from any L_p $(p_0 < p \leq \infty)$ to L_{r_0}, and from L_{p_0} to any L_r $(r < r_0)$.

Let $r < r_0$. Then from the boundedness of the operator A it follows that the r-th powers of the functions $\{Au(t)\}$, where $u(t)$ belong to some ball of the space L_{p_0}, have uniformly absolutely continuous integrals. This fact and Lemma 2 allow one to use Lemma 1, from which the second assertion of the theorem follows. To prove the first assertion it is necessary to pass to the adjoint operators.

Theorem 1, naturally, can also be applied in the study of operators with kernels that do not have the sign-constancy property. In this case one need only take into account the fact that the integral operator with kernel $K(t,s)$ is completely continuous if the operator with kernel $|K(t,s)|$ is continuous. The latter assertion, incidentally, can also be obtained as a consequence of Theorem 1. Moreover, from Lemmas 1 and 2 it follows directly that the operator A_1 with kernel $K_1(t,s)$ acts completely continuously from L_p to L_r, if $0 \leq K_1(t,s) \leq K(t,s)$, and the operator (1) acts from L_p to L_r completely continuously.

Let us now consider the frequently occurring case when (1) is a bounded operator acting from L_{p_0} to any L_r, $r < r_0$ (such, for example, are the operators of S. L. Sobolev of potential type, considered in [(4)]). From Theorem 1 it follows that the operator A is completely continuous. As applied to operators of potential type this means that the complete continuity of the embedding operators (Kondrashev' s theorem) is a consequence of Sobolev' s theorems on the boundedness of embedding operators.

3. In Theorem 1 it was shown that the operator (1), acting from L_{p_0} to L_{r_0}, is transformed into a completely continuous operator if either the space L_{p_0} is "narrowed" (one passes to L_p, where $p > p_0$), or the space L_{r_0} is "expanded" (one passes to L_r, where $r < r_0$). Here we shall show that the operator (1) is "transformed" into a completely continuous operator acting from L_{p_0} to L_{r_0}, if the singularities of the kernel $K(t,s)$ are "weakened" by an arbitrarily small amount.

We shall call a bounded function $\varphi(t,s)$ a **regularizer** of the kernel $K(t,s)$, if for every $\gamma > 0$ the integral operator

$$A_\gamma u(t) = \int_\Omega K(t,s)\,\varkappa(t,s;\gamma)\,u(s)\,ds \tag{6}$$

acts completely continuously from L_{p_0} to L_{r_0}, where $\varkappa(t,s;\gamma)$ denotes the characteristic function of the set of those points $\{t,s\} \in \Omega\times\Omega^*$ for which $|\varphi(t,s)| > \gamma$.

Theorem 2. Let a nonnegative kernel $K(t,s)$ possess a regularizer $\varphi(t,s)$, and let (1) be an operator acting from L_{p_0} to L_{r_0}. Then the linear integral operator

$$Bu(t) = \int_\Omega K(t,s)\varphi(t,s)u(s)\,ds \tag{7}$$

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also acts from L_{p_0} to L_{r_0} and is completely continuous.

This theorem follows directly from the following assertion of Zaanen.

Theorem 3 [9]. If a linear integral operator with a nonnegative kernel acts from L_{p_0} to L_{r_0}, then it is continuous.

As an example of the application of Theorem 2, let us again consider an operator of potential type with kernel $K(t,s) = \rho^{-\lambda}$. In [5,6] it is shown that it is bounded, but not completely continuous as an operator acting from L_p, where $p < \frac{n}{n-\lambda}$, to L_r, where

$$r = \frac{mp}{n - p(n-\lambda)}$$

(n and m are the dimensions of the domains Ω and Ω^*, respectively). It follows from Theorem 2 that operators with kernels $\rho^{-\lambda}|\ln\rho|^{\alpha}$, $\rho^{-\lambda}|\ln|\ln\rho||^{\alpha}$, etc., for any $\alpha < 0$, will act from L_p to L_r and will be completely continuous.

4. In proving Lemma 2 we showed that from the continuity of (1) as an operator from $L_p(\Omega)$ to $L_r(\Omega^*)$ there follows convergence, in the operator norm from $L_p(\Omega)$ to $L_1(\Omega^*)$, of the sequence A_n defined by equalities (5). With respect to this convergence one can make a number of additional assertions.

Lemma 3. The sequence of functions $A_{nu}(t)$, where $u(t)$ is an arbitrary fixed function from $L_p(\Omega)$, converges in $L_r(\Omega^*)$ to the function $Au(t)$ in norm.

For the proof it suffices to observe that, by Lemma 2, the sequence $A_{nu}(t)$ converges to $Au(t)$ in measure and that, moreover,

$$|A_{nu}(t)| \leq A|u(t)| \in L_r(\Omega^*).$$

Lemma 4. If (1) is a completely continuous operator acting from $L_p(\Omega)$ to $L_r(\Omega^*)$, then the sequence A_n converges to A in the norm of operators acting from $L_p(\Omega)$ to $L_r(\Omega^*)$.

Assuming the contrary, there would be a positive number ε_0 and a sequence of functions $u_n(t) \in L_p(\Omega)$ such that $\|u_n\| \leq 1$, but

$$\|(A - A_n)u_n(t)\|_{L_r(\Omega^*)} \geq \varepsilon_0 \qquad (n = 1, 2, \dots).$$

On the other hand, the sequence $(A - A_n)u_n(t)$ converges to zero in measure and $|(A - A_n)u_n(t)| \leq A|u_n(t)|$, whence follows the equiabsolutely continuity of the integrals of the r-th powers of the functions $(A - A_n)u_n(t)$; therefore $\|(A - A_n)u_n(t)\| \to 0$ as $n \to \infty$. The contradiction obtained proves the lemma.

The assertions of Lemmas 3 and 4 are obviously generalized to the case when one of the sets Ω and Ω^*, or both, have infinite measure.

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5. Lemma 4 makes it possible to obtain new simple criteria for complete continuity of the nonlinear integral operator of P. S. Urysohn

$$Tu(t) = \int_\Omega R[t, s, u(s)]\, ds. \tag{8}$$

Below it is assumed that the function $R(t, s, u)$ ($t \in \Omega^*$, $s \in \Omega$, $-\infty < u < \infty$) is continuous in the variable u and, for almost all $\{s, t\} \in \Omega \times \Omega^*$, is measur-

measurable in t and s for every u. In addition, it is assumed that

$$|R(t, s, u)| \leq K(t, s) f(s, u). \tag{9}$$

By f we shall denote the operator defined by the equality $fu(t) = f[t, u(t)]$ (see (7)), and by A the operator (1).

Theorem 4. *Let the operator f act from $L_{p_1}(\Omega)$ into $L_{p_3}(\Omega)$ and be continuous, and let the operator A act from $L_{p_3}(\Omega)$ into $L_{p_2}(\Omega^*)$ and be completely continuous, where $p_1, p_2 \geq 1$, $p_3 > 1$.*

Then the operator (6) acts from $L_{p_1}(\Omega)$ into $L_{p_2}(\Omega^*)$ and is completely continuous.

This theorem is a substantial strengthening of the results obtained in ([7,8]), and at the same time it relies on these results.

For the proof we introduce the sequence of operators

$$T_n u(t) = \int_\Omega R_n[t, s, u(s)]\, ds \quad (n = 1, 2, \ldots), \tag{10}$$

where

$$R_n(t, s, u) = \min\{|R(t, s, u)|;\ n f(s, u)\}\, \mathrm{sign}\, R(t, s, u).$$

It follows from Lemma 4 that the operators (10) converge uniformly on every ball to the operator (8). Therefore it remains only to show that each of the operators (10) acts from $L_{p_1}(\Omega)$ into $L_{p_2}(\Omega^*)$ and is completely continuous. From the results obtained in ([8]), it follows that the operators T_n act from $L_{p_1}(\Omega)$ into $L_{p_3}(\Omega^*)$ and are completely continuous. This completes the proof if $p_2 \leq p_3$. If, however, $p_2 > p_3$, then additional arguments are required: one must note that the set of values of the operators T_n in the p_2-th powers on every ball of the space $L_{p_1}(\Omega)$ has equi-absolutely continuous integrals, since the set of values on each such ball of the operator

$$Su(t) = \int_\Omega K(t, s) f[s, u(s)]\, ds$$

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is compact in $L_{p_2}(\Omega^*)$ and $|T_{nu}(t)| \leq Su(t)$; further, one must note that the set of values of each operator T_n is compact in measure and, finally, that the operators T_n transform every sequence of functions convergent in $L_{p_1}(\Omega)$ into a sequence convergent in measure.

6. The theorems presented in this article carry over without change to integral operators in spaces of vector-functions.

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