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Abstract

Full Text

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PRIMITIVELY π-FACTORIZABLE GROUPS

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A subgroup $\mathfrak{H}$ of a group $\mathfrak{G}$ is called **complemented** in $\mathfrak{G}$ if in $\mathfrak{G}$ there exists a subgroup $\mathfrak{K}$ such that $\mathfrak{G} = \mathfrak{H}\mathfrak{K}$ and $\mathfrak{H} \cap \mathfrak{K} = 1$. It is known that a group $\mathfrak{G}$ in which all its subgroups are complemented (such a group is called completely factorizable) is a semidirect product of cyclic subgroups of prime orders (see ([1])). It is also known that a finite group in which all its Sylow p-subgroups are complemented is solvable (see P. Hall ([2])). Thus, the assumption of the existence in a group of one or another system of complemented subgroups makes it possible to draw definite conclusions concerning its structure. In this connection, S. N. Chernikov (see ([3])) posed the general problem of studying groups with one or another previously prescribed system of complemented subgroups. In the author' s papers ([4–6]), in particular, groups were studied (mainly periodic ones) whose system of complemented subgroups coincides with the totality of all cyclic subgroups of prime orders. In the present note this system is narrowed to the system of cyclic subgroups whose orders belong to some fixed set of prime numbers π (such groups were called primitively π-factorizable in paper ([4])).

1. A completely factorizable group $\mathfrak{A}$ will be called completely primitive if there exists at least one such cyclic group of prime order, dividing the order of the group $\mathfrak{A}$, whose holomorph contains a subgroup isomorphic to the group $\mathfrak{A}$.

A group $\mathfrak{A}$ with completely primitive Sylow π-subgroups (π is a fixed set of prime numbers) will be called π-primitive if it is isomorphic to a subgroup of the holomorph of some cyclic group of prime order dividing the order of the group $\mathfrak{A}$ and contained in the set π. An element H of prime order $p \in \pi$ of an arbitrary group $\mathfrak{H}$ will be called π-primitive if there exists such a normal divisor $\mathfrak{N}$ of the group $\mathfrak{H}$ not containing it that the factor group $\mathfrak{H}/\mathfrak{N}$ is π-primitive.

Lemma 1. *If $\mathfrak{N}$ is some normal divisor of a periodic primitively π-factorizable group $\mathfrak{H}$, then it contains its π-primitive elements if and only if there exists a normal divisor $\mathfrak{M}$ of the group $\mathfrak{H}$ such that the factor group $\mathfrak{N}\mathfrak{M}/\mathfrak{M}$ is π-primitive.*

Theorem 1. *A periodic group is primitively π-factorizable if and only if it is an extension of some primitively π-factorizable group containing no π-primitive elements by means of a subgroup of some complete direct product of π-primitive groups (each π-element of a product of this kind is π-primitive).*

Corollary. *A locally solvable periodic group is primitively π-factorizable if and* *only if it is an extension of a group containing no π-elements by means of a subgroup of some complete direct product of π-primitive groups.*

For primitively π-factorizable groups containing no π-primitive elements, the following two theorems are valid.

Theorem 2. *A periodic primitively π-factorizable group containing no π-primitive elements has a decreasing invariant series whose factors containing π-elements are simple nonabelian primitively π-factorizable groups.*

Theorem 3. An arbitrary group possessing an invariant system whose factors containing π-elements are simple nonabelian primitive π-factorizable groups is primitively π-factorizable and has no π-primitive elements.

Corollary. Every π'-subgroup ($\pi' \subseteq \pi$) of a group satisfying the condition of Theorem 3 is locally finite.

Let us note that any simple nonabelian primitively π-factorizable group containing π-elements is finite (see Theorem 3 of ([4])). Its order is divisible by only one prime number $p \in \pi$. Moreover, it is isomorphic to some subgroup of the group of substitutions of p symbols.

2. If some group is primitively π-factorizable, then all its subgroups have the same property, but, generally speaking, not all factor groups do (see ([6])).

For periodic groups the following holds.

Theorem 4. A homomorphic image of a periodic primitively π-factorizable group is primitively π-factorizable.

This theorem allows one to prove the following theorem:

Theorem 5. A periodic group is primitively π-factorizable if and only if it is an extension of a group containing no π-elements by a subgroup of the complete direct product of finite primitively π-factorizable groups.

The necessity of the condition of this theorem was established in ([4]).

3. Let $\mathfrak{A}$ be such a subgroup of the complete direct product of groups $\mathfrak{A}_\alpha$ ($\alpha \in \mathfrak{M}$) that its component in each of the groups $\mathfrak{A}_\alpha$ coincides with the latter. We shall call the group $\mathfrak{A}$ the π-subdirect product of the groups $\mathfrak{A}_\alpha$ (π is a fixed set of prime numbers) if every one of its π-elements has a finite number of components distinct from the identity.

By virtue of this definition, every subgroup of a π-subdirect product is again a π-subdirect product.

We shall call a group $\mathfrak{G}$ π-locally normal if every one of its π-elements is contained in its finite normal divisor. Obviously, the π-subdirect product of finite groups is a π-locally normal group. The converse assertion is false even for locally normal groups. Let us note that the subgroup generated by all π-elements of a π-locally normal group may be nonisomorphic to a subgroup of a direct product of finite groups.

Example. Let the groups $\mathfrak{G}_n$ $(n = 1, 2, \ldots)$ be nonsplit extensions of the quaternion group by a group of third order. Denote by A_n $(n = 1, 2, \ldots)$ an element of second order of the group $\mathfrak{G}_n$, lying in its center. If $\mathfrak{G} = \prod_n \mathfrak{G}_n$ is the direct product of the groups $\mathfrak{G}_n$ and $\mathfrak{N}$ is its subgroup generated by the elements $A_{k A} l$ $(k, l = 1, 2, \ldots)$, then the factor group cannot be isomorphically embedded in a direct product of finite groups. Taking as the set π the set consisting of the single number 3, we obtain that the group $\mathfrak{G}/\mathfrak{N}$ is π-locally normal and is generated by elements of order 3.

In the present note, conditions are given under which the subgroup generated by all π-elements of a π-locally normal group is isomorphic to some subgroup of a direct product of finite groups.

Theorem 6. The subgroup generated by all π-elements of a periodic finitely approximable π-locally normal group without center is isomorphic to a subgroup of some direct product of finite groups.

A group $\mathfrak{G}$ is called finitely approximable if, for any of its elements, there exists a normal divisor of finite index in $\mathfrak{G}$ not containing it (see ([8])).

This generalization of the theorem from ([7]) is nontrivial, since the subgroup under consideration may have a center of arbitrary cardinality.

Corollary. The subgroup generated by all the π-elements of a periodic finitely approximable primitive π-factorizable π-locally normal group whose center is a π-subgroup is a subgroup of some direct product of finite primitive π-factorizable groups.

The corollary follows from the preceding theorem by virtue of the following lemma.

Lemma 2. In a periodic primitive π-factorizable group, every π-subgroup of its center is complemented.

It is known that a locally normal group without center is isomorphic to a subgroup of some direct product of finite groups (see (7); for countable groups this was established by Ph. Hall (9)). The following theorem serves as a generalization of this proposition.

Theorem 7. If a periodic finitely π-approximable group $\mathfrak{G}$ is π-locally normal and its center contains no π-elements, then it is an extension of a group containing no π-elements by a π-subdirect product of finite groups.

We shall call a group $\mathfrak{G}$ finitely π-approximable if, for every π-element of it, there exists a normal divisor of finite index not containing this element.

Corollary. A primitive π-factorizable π-locally normal periodic group is an extension of a group containing no π-elements by some π-subdirect product of finite primitive π-factorizable groups.

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Note: Figure translations are in progress. See original paper for figures.