THEORY OF ELASTICITY

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Abstract

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ON A PROBLEM OF THE THEORY OF ELASTICITY FOR AN INFINITE STRIP IN DISPLACEMENTS

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In our papers [1–3] it was shown that, by applying the concept of "conjugate-on-the-contour functions," one can obtain a very effective general solution of the problem for an infinite strip in stresses. In connection with the publication of an article [4*] on this same subject, we shall show here that the method we used for solving the problem in stresses gives an analogous solution of the problem in displacements as well.

Consider the infinite strip ($-a \le y \le a$; $-\infty < x < \infty$). We write the well-known equations of G. V. Kolosov in the following form:

$$\begin{gathered} 2\mu(u - iv) = \frac{2k-1}{2}\overline{\Phi(z)} - \frac{1}{2}\Phi(z) + iy\Phi'(z) - a\Psi(z); \\ X_x + Y_y = 2R\Phi'(z); \qquad Y_y - X_x + 2iX_y = 2a\Psi'(z) - 2iy\Phi''(z), \end{gathered} \tag{1}$$

where $z = x + iy$; $k = 2(1-\sigma)$ for plane strain; $k = \dfrac{2}{1+\sigma}$ for plane stress; σ is Poisson' s ratio. It is required to find $\Phi(z)$ and $\Psi(z)$ satisfying the boundary conditions.

By functions conjugate on the contour we mean two functions of the complex variable $P(z)$ and $Q(z)$, taking real values on the real axis, such that on the contour of the strip

$$\begin{aligned} P(x + ia) &= r(x) + ip(x), \\ Q(x + ia) &= q(x) + ir(x). \end{aligned} \tag{2}$$

Let us consider four cases of the deformed state of the strip.

1. Put

$$\Phi(z) = P(z); \qquad \Psi(z) = Q'(z) - \frac{k}{a}P(z); \tag{3}$$

then, in accordance with (1) and (2), on the contour of the strip ($y = \pm a$) we shall have

$$2\mu u = (2k-1)r(x) - aq'(x) - ap'(x), \qquad v = 0. \tag{4}$$

2. Let

$$\Phi(z) = Q(z); \qquad \Psi(z) = -P'(z) + \frac{k-1}{a}Q(z); \tag{5}$$

then for $y = \pm a$

$$u = 0; \qquad 2\mu v = \pm[(2k-1)r(x) - aq'(x) - ap'(x)]. \tag{6}$$

3. If

$$\Phi(z) = -iQ(z); \qquad \Psi(z) = iP'(z) + \frac{ik}{a}Q(z), \tag{7}$$

* We note, incidentally, that for the existence of a general solution of the problem for an infinite strip in stresses it is sufficient that only one statics condition be satisfied:

$$\int_{-\infty}^{+\infty} N(t)\,dt = 0,$$

where $N(x)$ are the normal components of the stresses on the contour of the strip (see [3]). The fulfillment of the other two statics conditions, as required in article [4], is superfluous.

then on the contour

$$2\mu u = \pm\left[(2k-1)r(x) + aq'(x) + ap'(x)\right]; \qquad v = 0. \tag{8}$$

4. If, however,

$$\Phi(z) = iP(z); \qquad \Psi(z) = iQ'(z) + \frac{i(k-1)}{a}P(z), \tag{9}$$

then on the contour

$$u = 0; \qquad 2\mu v = (2k-1)r(x) + aq'(x) + ap'(x). \tag{10}$$

Thus, for symmetric deformation of the strip (cases 1 and 2), the problem reduces to finding two conjugate contour functions satisfying the condition

$$(2k-1)r(x) - aq'(x) - ap'(x) = 2\mu f(x), \tag{11}$$

while for antisymmetric deformation of the strip (cases 3 and 4), satisfying the condition

$$(2k-1)r(x) + aq'(x) + ap'(x) = 2\mu f(x), \tag{12}$$

where $f(x)$ is the prescribed displacement on the contour of the strip.

In paper ([1]), by applying the Schwarz integral for an infinite strip, we found that*

$$P'(z) + iQ'(z) = \frac{im}{4a}\int_{-\infty}^{+\infty} \frac{r(t)\,dt}{\operatorname{ch}^2\frac{m}{2}(t-z+ia)}, \qquad m = \frac{\pi}{2a}, \tag{13}$$

and then from (2) and (13) it follows that

$$ap'(x) + aq'(x) = \frac{m}{4}\int_{-\infty}^{+\infty} \frac{r(t)\,dt}{\operatorname{ch}^2\frac{m}{2}(t-x)}. \tag{14}$$

Thus, (11) can be rewritten in the form of the following integral equation

$$(2k-1)r(x) - \frac{m}{4}\int_{-\infty}^{+\infty} \frac{r(t)\,dt}{\operatorname{ch}^2\frac{m}{2}(t-x)} = 2\mu f(x). \tag{15}$$

Hence, assuming that $f(x)$ is absolutely integrable and satisfies the Dirichlet conditions, we obtain

$$r(x) = \frac{\mu}{\pi}\int_{-\infty}^{+\infty} \frac{\operatorname{sh} 2a\xi\,d\xi}{(2k-1)\operatorname{sh} 2a\xi - 2a\xi}\int_{-\infty}^{+\infty} f(t)\cos\xi(t-x)\,dt. \tag{16}$$

Substituting (16) into (13), we find (up to a constant) that

$$P(z) + iQ(z) = -\frac{2i\mu}{\pi}\int_{-\infty}^{+\infty} \frac{d\xi}{(2k-1)\operatorname{sh} 2a\xi - 2a\xi}\int_{-\infty}^{+\infty} f(t)\sin\xi(t-z+ia)\,dt \tag{17}$$

for symmetric deformation of the strip.

In exactly the same way, from (12), (14), and (13), we find that

$$P(z)+iQ(z)=-\frac{2i\mu}{\pi}\int_{-\infty}^{+\infty}\frac{d\xi}{(2k-1)\operatorname{sh}2a\xi+2a\xi}\int_{-\infty}^{+\infty}f(t)\sin\xi(t-z+ia)\,dt \tag{18}$$

for antisymmetric deformation of the strip.

* The combined notation $P(z)+iQ(z)$ greatly simplifies the subsequent calculations. Since these functions are real on the real axis, their separation is carried out in the usual way.

If, however, $f(x)$ is represented by a Fourier series

$$f(x)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}(A_n\cos\gamma x+B_n\sin\gamma x),\qquad \gamma=\frac{n\pi}{l},$$

then, by a method analogous to that described, we obtain

$$P(z)+iQ(z)=\frac{A_0\mu}{2a(k-1)}(a+iz)+4i\mu\sum_{n=1}^{\infty}\frac{A_n\sin\gamma(z-ia)-B_n\cos\gamma(z-ia)}{(2k-1)\operatorname{sh}2a\gamma-2a\gamma} \tag{19}$$

for symmetric deformation of the strip, and

$$P(z)+iQ(z)=\frac{A_0\mu}{2ak}(a+iz)+4i\mu\sum_{n=1}^{\infty}\frac{A_n\sin\gamma(z-ia)-B_n\cos\gamma(z-ia)}{(2k-1)\operatorname{sh}2a\gamma+2a\gamma} \tag{20}$$

for antisymmetric deformation of the strip.

Moreover, without dwelling here on the proof, we note that, on the basis of the results obtained above, the solution of the problem in polynomials can be represented in the following form.

For symmetric deformation of the strip ($f(x)$ is a polynomial),

$$P(z)+iQ(z)=\frac{2i\mu}{k-1}\left[b_{-1}h^{-1}\int_0^{z-ia}f(t)\,dt+b_1hf'(z-ia)+\right.$$
$$\left.+b_3h^3f^{(3)}(z-ia)+b_5h^5f^{(5)}(z-ia)+...\right], \tag{21}$$

where $h=2a$; $b_{-1},b_1,b_3,...$ are the coefficients of the expansion of

$$\frac{1}{(2k-1)\operatorname{sh}x-x}$$

in powers of x,

$$\left[b_{-1}=1,\quad b_1=\frac{\alpha}{6},\quad b_3=\frac{\alpha}{12}\left(\frac{\alpha}{3}-\frac{1}{10}\right),\quad b_5=\frac{\alpha}{72}\left(\frac{\alpha^2}{3}-\frac{\alpha}{5}+\frac{1}{70}\right),\ldots;\ \alpha=\frac{2k-1}{2(k-1)}\right].$$

For antisymmetric deformation of the strip,

$$P(z)+iQ(z)=\frac{2i\mu}{k}\left[c_{-1}h^{-1}\int_0^{z-ia}f(t)\,dt+c_1hf'(z-ia)+c_3f^{(3)}(z-ia)+\right.$$

$$\left.+c_5h^5f^{(5)}(z-ia)+\ldots\right], \qquad (22)$$

where $c_{-1},c_1,c_3,\ldots$ are the coefficients of the expansion of

$$\frac{1}{(2k-1)\operatorname{sh}x+x}$$

in a power series,

$$\left[c_{-1}=1,\quad c_1=\frac{\beta}{6},\quad c_3=\frac{\beta}{12}\left(\frac{\beta}{3}-\frac{1}{10}\right),\quad c_5=\frac{\beta}{72}\left(\frac{\beta^2}{3}-\frac{\beta}{5}+\frac{1}{70}\right),\ldots;\ \beta=\frac{2k-1}{2k}\right].$$

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4. S. M. Belonosov, DAN, 131, No. 6 (1960).

Note: Figure translations are in progress. See original paper for figures.