MATHEMATICS

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Abstract

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CONSTRUCTION OF IRREDUCIBLE REPRESENTATIONS OF SIMPLE ALGEBRAIC GROUPS OVER A FINITE FIELD

1. As is known ([5,6,1]), the irreducible unitary representations of any complex simple Lie group G can be obtained by the following construction. One takes the homogeneous space $X = G/Z$ of right cosets of the group G by its maximal nilpotent subgroup Z. In the space of functions $f(x)$ on X one considers the representation $T(g)f(x) = f(xg)$. Decomposing $T(g)$ into irreducible representations, we obtain all irreducible representations of the "principal series" (i.e., those contained in the regular representation of the group G). In passing to simple algebraic groups over other fields, such a construction no longer gives all, but only a small part, of the irreducible representations.

In the present note a construction and classification are given of the irreducible representations of the simple algebraic groups of Dickson–Chevalley over a **finite** field ([3,4]). By Chevalley' s construction, each of these groups is determined by specifying a simple Lie algebra over the field of complex numbers and a finite field K. In particular, to the classical simple Lie algebras there correspond the following infinite series of simple groups: a) the factor group of the group of unimodular matrices of order n (with entries in K) by its center; b) the commutant of the orthogonal projective group (with entries in K); c) the factor group by the center of the symplectic group (with entries in K).

2. The construction and classification of all irreducible representations of the Dickson–Chevalley groups are based on the following two basic theorems.

Theorem 1. Let G be a Dickson–Chevalley group, and Z its maximal nilpotent subgroup. Consider all possible characters $\chi(\xi)$, i.e., one-dimensional representations of the subgroup Z. To each character χ we associate the representation of the group G induced by it. (This representation is realized in the space of functions $f(g)$ on G such that $f(\xi g) = \chi(\xi)f(g)$ for every $\xi \in Z$. The operator of the representation has the form $T_\chi(g_0)f(g) = f(gg_0)$.) It is asserted that every irreducible representation of the group G is contained in at least one of the representations $T_\chi(g)$.

Equivalent formulation of Theorem 1. In any representation $T(g)$ of the group G there exists a vector that is proper for all operators $T(\xi)$, $\xi \in Z$.

We now introduce on the set of characters $\chi(\xi)$ a partial ordering: we shall say that $\chi_1 < \chi_2$ if, on every root subgroup in Z where $\chi_2(\xi) \equiv 1$, we also have $\chi_1(\xi) \equiv 1$. Maximal characters χ will be called characters of **general position**.

Theorem 2. Let χ be a character of general position. Then every irreducible representation of the group G is contained in the representation $T_\chi(g)$ with multiplicity not greater than 1.

Equivalent formulation of Theorem 2. If χ is a character of general position, then in any irreducible representation $T(g)$ of the group G there is no more than one proper vector with respect to the operators $T(\xi)$, $\xi \in Z$, with proper value $\chi(\xi)$.

Theorems 1 and 2 replace Cartan' s theory of highest weights, which is not applicable to the groups under consideration*.

For simplicity we shall give the proof of Theorems 1 and 2 only for the group G_n of unimodular matrices of order n over the field K. The maximal nilpotent subgroup Z of the group G_n consists of triangular matrices $\|\zeta_{ij}\|$, where $\zeta_{ii} = 1,\ \zeta_{ij} = 0$ for $i > j$.

Proof of Theorem 1. It suffices to prove the theorem for representations of the subgroup $G'_n \subset G_n$ of matrices $\|g_{ij}\|$ for which $g_{11} = 1,\ g_{i1} = 0$ for $i > 1$. We shall carry out the proof by induction on n, assuming the theorem already proved for $n-1$. (For $n = 2$ the theorem is obvious.) Let $T(g)$ be a representation of G'_n acting in some space H. Consider in G'_n the subgroup Z'_n of matrices $\|\zeta_{ij}\|$ for which $\zeta_{ij} = 0$ when $i > 1,\ i \neq j,\ \zeta_{ii} = 1$. This subgroup is commutative, and therefore H decomposes into vectors that are eigenvectors with respect to the operators $T(\zeta')$, $\zeta' \in Z'_n$, with eigenvalues

$$\chi(\zeta') = \chi_1(\zeta_{12}) \cdots \chi_{n-1}(\zeta_{1n})$$

(χ_i are additive characters on K). The characters χ may be regarded as points of an $(n-1)$-dimensional linear space over K. The subgroup Z'_n is a normal divisor in G'_n, and therefore the operators $T(g)$, $g \in G'_n$, carry these eigenvectors again into eigenvectors; thus the elements $g \in G'_n$ define transformations on the set of characters $\chi(\zeta')$. It is easy to see that by these transformations any $\chi(\zeta')$ can be carried into some $\chi_0(\zeta')$ independent of $\zeta_{13}, \dots, \zeta_{1n}$. Let H_{χ_0} be the subspace of vectors with eigenvalue $\chi_0(\zeta')$. This subspace is invariant with respect to the operators $T(g')$, where g' runs through the subgroup $G''_n \simeq G'_{n-1}$ of matrices from G'_n for which $g_{1i} = 0$ when $i > 1,\ g_{22} = 1,\ g_{j2} = 0$ when $j > 2$. Consequently, by the induction hypothesis, in H_{χ_0} there exists a vector which is an eigenvector for all $T(\xi)$, $\xi \in Z \cap G''_n$. Obviously, this vector will also be an eigenvector for all operators $T(\xi)$, $\xi \in Z$. Theorem 1 is proved. For the other classes of Dickson–Chevalley groups the proof is carried out differently.

Proof of Theorem 2. We need to prove that the ring R_χ of operators commuting with the operators $T_\chi(g)$, where χ is a character in general position, is commutative (this is an equivalent formulation of Theorem 2). The ring R_χ

is isomorphic to the ring of all functions $f(g)$ on the group satisfying, for all $\zeta_1, \zeta_2 \in Z$, the relation

$$f(\zeta_1 g \zeta_2) = \chi(\zeta_1) f(g) \chi(\zeta_2),$$

with convolution as multiplication. Any matrix from G_n can be represented in the form $g = \zeta_1 \delta s \zeta_2$, where $\zeta_1, \zeta_2 \in Z$, δ is a diagonal matrix (i.e. a matrix from the Cartan subgroup), and s is a permutation matrix (i.e. an element of the Weyl group). Therefore it is sufficient to specify functions $f \in R_\chi$ only on the matrices δs. Without loss of generality one may assume that the character in general position χ has the form

$$\chi(\zeta) = \chi_1(\zeta_{12} + \cdots + \zeta_{n-1,n}),$$

where χ_1 is an additive character on K. Let $\mathfrak{M}$ be the set of matrices δs on which the functions $f \in R_\chi$ can take values different from 0. It is easy to verify that the matrices $\delta s \in \mathfrak{M}$ satisfy the condition $(\delta s)^\theta = \delta s$, where θ is an involutive antiautomorphism of the group G_n preserving Z**. Take as a basis in R_χ the functions $f_{\delta s}$ (where $\delta s \in \mathfrak{M}$) defined by the following conditions: $f_{\delta s}(g) = 1$ when $g = \delta s$, and $f_{\delta s}(\delta_1 s_1) = 0$ when $\delta_1 s_1 \neq \delta s$. It can be shown that the convolution $f(g) = f_{\delta_1 s_1} \circ f_{\delta_2 s_2}$ of two such functions is expressed by the following formula:

$$f(\delta s) = \frac{N}{N_{s_1} N_{s_2}} \sum \chi(\zeta)\chi(\zeta_1)\chi(\zeta_2),$$

* In the case of the classical complex Lie groups, the integral-form theory of Cartan' s highest weights is the construction of Gelfand and Naimark (1). This construction is a special case of the construction given in the present note, for $\chi \equiv 1$.

** This antiautomorphism has the form $g^\theta = s_0 g' s_0$, where the prime denotes transposition, and s_0 is the permutation matrix whose ones stand on the secondary diagonal.

where the sum is taken over all possible triples of matrices ζ, ζ_1, ζ_2 from Z, connected with one another by the relation $\delta s = \zeta_1 \delta_1 s_1 \zeta \delta_2 s_2 \zeta_2$; N is the order of the subgroup Z, N_{s_i} is the order of the subgroup $s_i Z s_i^{-1} \cap Z$. The commutativity of the ring R_χ, i.e., the fact that $f_{\delta_1 s_1} \circ f_{\delta_2 s_2} = f_{\delta_2 s_2} \circ f_{\delta_1 s_1}$, follows directly from this formula. For this it suffices to observe that, by virtue of the condition imposed on the matrices $\delta s \in \mathfrak{M}$, the equality $\delta s = \zeta_1 \delta_1 s_1 \zeta \delta_2 s_2 \zeta_2$ is equivalent to the equality $\delta s = \zeta_2^0 \delta_2 s_2 \zeta^0 \delta_1 s_1 \zeta_1^0$; and, by the assumption made about the character χ, we have $\chi(\zeta^0) = \chi(\zeta)$ for any $\zeta \in Z$. For the other Dickson-Chevalley groups the proof of Theorem 2 is carried out analogously.

3. An irreducible representation of a Dickson-Chevalley group G will be called **principal** if it contains a proper vector of general position with respect to the subgroup Z; a representation will be called **degenerate**

if it has no such vector. It can be shown that the number of principal representations is expressed as a polynomial of degree l in the order k of the ground field (l is the dimension of the Cartan subgroup); whereas the number of degenerate representations is expressed by a polynomial of lower degree. In this sense the principal representations are almost all irreducible representations of the group G.* In order to construct the principal representations of the group G, we must decompose the representation $T_\chi(g)$, corresponding to the character χ of general position, into irreducible representations. Since the ring R_χ of operators commuting with $T_\chi(g)$ is commutative, the problem reduces to finding in the ring R_χ such functions φ that $\varphi \circ \psi = c_\psi \varphi$ for every $\psi \in R_\chi$. It can be shown that these functions φ are determined from the functional relation

$$\varphi(\delta_1 s_1)\varphi(\delta_2 s_2) = \lambda \sum \chi(\zeta^{-1})\chi(\zeta_1^{-1})\chi(\zeta_2^{-1})\varphi(\delta s),$$

where the sum is taken over all matrices $\delta s \in \mathfrak{M}$, and over $\zeta, \zeta_1, \zeta_2 \in Z$, connected with one another by the relation $\delta s = \zeta_1 \delta_1 s_1 \zeta \delta_2 s_2 \zeta_2$. The solutions of this equation—the functions $\varphi(\delta s)$—we shall call Bessel functions associated with the group G. For the group of unimodular matrices of the second order these functions have been found explicitly in ([2]).

4. The irreducible representations of the group G split into series. We assign two representations to the same series if, for each $\chi(\zeta)$, $\zeta \in Z$, in these representations there corresponds the same number of linearly independent proper vectors (or, in other words, if these representations occur in the same $T_\chi(g)$ and with the same multiplicity). As an example, we indicate the series of irreducible representations of the group G_3 of unimodular matrices of the third order (in the case when the number $k-1$ is not divisible by 3).

The group G_3 has three series of principal representations "of general position" : 1) $\frac{1}{6}(k-2)(k-3)$ representations of dimension $(k+1)(k^2+k+1)$; they occur in all $T_\chi(g)$, and in $T_\chi(g)$, $\chi \equiv 1$, they occur with multiplicity 6, while in $T_\chi(g)$, $\chi \neq 1$, χ_0 (χ_0 is a character of general position), with multiplicity 3; 2) $\frac{1}{2}k(k-1)$ representations of dimension k^3-1; they occur only in $T_\chi(g)$, $\chi \neq 1$, each with multiplicity 1; 3) $\frac{1}{3}k(k+1)$ representations of dimension $(k-1)(k^2-1)$; they occur only in $T_{\chi_0}(g)$.

There are also two "special" series of principal representations: 4) $k-2$ representations of dimension $k(k^2+k+1)$, and 5) one representation of dimension k^3. Finally, there are three series of degenerate representations: 6) $k-2$ representations of dimension k^2+k+1; 7) one representation of dimension k^2+k, and 8) one unit representation (cf. ([7]), where the characters of the irreducible representations of the group G_3 are computed).

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* We note that the dimension of a principal representation is always a polynomial in k of degree N (N is the dimension of the subgroup Z); the dimension of any degenerate representation is expressed by a polynomial in k of lower degree.

In the case of the group of unimodular matrices of order n, each series of basic representations of "general position" is specified by a partition $n = n_1 + \dots + n_r$ of the number n into a sum of positive integers. It can be shown that the dimensions of the representations in this series are equal to

$$\frac{(k-1)(k^2-1)\dots(k^n-1)}{(k^{n_1}-1)\dots(k^{n_r}-1)}$$

(cf. ([8])).

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REFERENCES

[1] I. M. Gelfand, M. A. Naimark, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **36** (1950).
[2] I. M. Gelfand, M. I. Graev, *DAN*, **146**, No. 4 (1962).
[3] C. Chevalley, *Sborn. per. Matematika*, **2**, No. 1, 1958.
[4] L. E. Dickson, *Linear Groups*, N. Y., 1958.
[5] I. M. Gelfand, M. A. Naimark, *DAN*, **54**, 195 (1946).
[6] I. M. Gelfand, M. A. Naimark, *Matem. sborn.*, **21** (**63**), No. 3, 405 (1947).
[7] R. Steinberg, *Canad. J. Math.*, **3**, No. 2, 225 (1951).
[8] J. A. Green, *Trans. Am. Math. Soc.*, **80**, 402 (1955).

Note: Figure translations are in progress. See original paper for figures.