Hydromechanics

1962

SovietRxiv

Abstract

Full Text

Hydromechanics

G. S. Golitsyn

Fluctuations of Dissipation in a Locally Isotropic Turbulent Flow

(Presented by Academician A. N. Kolmogorov, 5 I 1962)

1. In the theory of locally isotropic turbulence developed by A. N. Kolmogorov ([1]), the mean energy dissipation $\bar{\varepsilon}$ plays the fundamental role. The dissipation ε, referred to unit mass of a viscous incompressible fluid, is related to the gradients of the velocity field by the well-known formula:

$$\varepsilon = \frac{1}{2}\nu\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)^2 = \nu\left[\left(\frac{\partial v_i}{\partial x_k}\right)^2 + \frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_i}\right], \tag{1}$$

where ν is the kinematic viscosity of the medium. Summation over repeated indices is understood. In a turbulent flow the velocity gradients are random quantities; therefore the dissipation of the kinetic energy of turbulence will also be a random function of the spatial coordinates and time. It is of interest to investigate the statistical structure of the dissipation field. In the present note, on the basis of Kolmogorov' s theory and with some additional hypotheses, we calculate the correlation function of dissipation fluctuations and its spectrum, as well as the variance of the dissipation averaged over space as a function of the magnitude of the averaging volume. The correlation function of dissipation fluctuations is needed, in particular, for calculating fluctuations of the energy dissipation averaged over a certain sphere, and also for resolving the question of the degree of accuracy with which statistical averaging (over the ensemble) may be replaced by spatial averaging under conditions of locally isotropic turbulence. The calculation of the variance of the averaged dissipation as a function of the magnitude of the averaging volume was undertaken in connection with recent works by A. M. Obukhov ([2]) and A. N. Kolmogorov ([3]), which proposed a new construction of the theory of locally isotropic turbulence that at once takes into account the random character of energy dissipation. In these works a related characteristic appears: the variance of the logarithm of the dissipation averaged over a certain sphere, as well as the variance of this dissipation itself.

2. The correlation function of the fluctuations of dissipation ε at points 1 and 2 of a locally isotropic turbulent flow will depend only on the distance r between these two points:

$$\overline{(\varepsilon_1 - \bar{\varepsilon})(\varepsilon_2 - \bar{\varepsilon})} = B_{\varepsilon\varepsilon}(r),$$

where $\bar{\varepsilon}$ denotes the mean dissipation. Since the dissipation is quadratic with respect to the velocity gradients, the correlation function $B_{\varepsilon\varepsilon}(r)$ will be determined by fourth moments of the gradients. In order to express the unknown fourth moments through second moments, we shall use Millionshchikov' s hypothesis that the fourth moments are related to the second moments in the same way as for a normal distribution. In that case the calculations lead to the following expression for the correlation function:

$$B_{\varepsilon\varepsilon}(r) = \frac{\nu^2}{2}\frac{\partial^2 D_{ij}}{\partial r_k \partial r_l}\left(\frac{\partial^2 D_{ij}}{\partial r_k \partial r_l} + \frac{\partial^2 D_{jk}}{\partial r_i \partial r_l} + \frac{\partial^2 D_{il}}{\partial r_j \partial r_k} + \frac{\partial^2 D_{kl}}{\partial r_i \partial r_j}\right), \tag{2}$$

where the structural tensor of the velocity field D_{ij} is expressed in the known manner through the longitudinal and transverse velocity structure functions:

$$D_{ij}(r) = [D_{ll}(r) - D_{nn}(r)](r_i r_j / r^2) + D_{nn}(r)\delta_{ij}.$$

Using this formula and carrying out the differentiation and summation in formula (2), after elementary but cumbersome transformations we obtain:

$$\begin{aligned}\nu^{-2}B_{\varepsilon\varepsilon}(r) &= 2D_{ll}''^2 + D_{nn}''^2 + \frac{2}{r}D_{nn}''(3D_{ll}' - 2D_{nn}') \\ &+ \frac{1}{r^2}\left(17D_{ll}'^2 - 28D_{ll}'D_{nn}' + 18D_{nn}'^2\right) \\ &- \frac{2(D_{ll} - D_{nn})}{r^2}\left[4D_{nn}'' + \frac{20}{r}D_{ll}' - \frac{26}{r}D_{nn}' - \frac{23}{r^2}(D_{ll} - D_{nn})\right].\end{aligned} \tag{3}$$

The theory of locally isotropic turbulence makes it possible to determine the form of the velocity structure functions at scales much smaller and much larger than the inner scale of turbulence $\lambda_0 = (\nu^3/\bar{\varepsilon})^{1/4}$. According to ([4])

$$\text{for } r \ll \lambda_0, \quad D_{ll}(r) = \frac{1}{15}\frac{\bar{\varepsilon}}{\nu}r^2, \quad D_{nn}(r) = \frac{2}{15}\frac{\bar{\varepsilon}}{\nu}r^2,$$

$$\text{for } \lambda_0 \ll r \ll L, \quad D_{ll}(r) = \frac{3}{4}C(\bar{\varepsilon}r)^{2/3}, \quad D_{nn}(r) = C(\bar{\varepsilon}r)^{2/3}.$$

Here C is a certain numerical constant determined from experiments, and L is the outer scale of turbulence. With the aid of these formulas one can find the mean square of the fluctuations of ε, as well as the asymptotic behavior of the correlation function $B_{\varepsilon\varepsilon}(r)$ for $\lambda_0 \ll r \ll L$, in the so-called inertial range of scales.

Substituting into (3) the values of the structure functions for $r \ll \lambda_0$, we obtain:

$$B_{\varepsilon\varepsilon}(0) = \overline{(\varepsilon - \bar{\varepsilon})^2} = \overline{\varepsilon^2} - \bar{\varepsilon}^2 = \frac{2}{5}\bar{\varepsilon}^2, \tag{4}$$

whence

$$\overline{\varepsilon^2} = 1.4\,\bar{\varepsilon}^2, \qquad \sigma_\varepsilon/\bar{\varepsilon} = \sqrt{0.4} \simeq 0.632. \tag{5}$$

In the other limiting case, i.e., in the inertial interval, we obtain:

$$B_{\varepsilon\varepsilon}(r) = \frac{1109}{648}\,C^2\nu^2\bar{\varepsilon}^{4/3}r^{-8/3} \simeq 1.71\,C^2\nu^2\bar{\varepsilon}^{4/3}r^{-8/3}. \tag{6}$$

As to the magnitude of the correlation radius of dissipation, we can say only that it must be of the order of the inner scale of turbulence λ_0. In order that the function $B_{\varepsilon\varepsilon}(r)$ could be determined for all r from the equilibrium interval $r \ll L$, additional hypotheses are necessary. We shall here assume that the spectral density of the energy of turbulence in the equilibrium interval has the form

$$E(k) = \frac{a^{2/3}}{\Gamma(2/3)}\,\bar{\varepsilon}^{2/3}k^{-5/3}e^{-a(k\lambda_0)^2}, \tag{7}$$

where k is the wave number and a is a certain constant that can be related to the structure constant C. It turns out that $a \simeq 0.685\,C^{3/2}$. A known justification of this hypothesis is provided by the work of E. A. Novikov ([5]). In the case of a spectral density of this form, one can also find explicit formulas for the velocity structure functions for all values $r \ll L$. With the aid of these formulas, for the correlation coefficient of the dissipation fluctuations, the expression can be obtained

$$\begin{aligned} b_{\varepsilon\varepsilon}(x) = \frac{B_{\varepsilon\varepsilon}(r/\sqrt{a}\lambda_0)}{B_{\varepsilon\varepsilon}(0)} &= \frac{5}{2}M_{3/2}^2 + \frac{235}{18}M_{5/2}^2 + \frac{56}{9}M_{7/2}^2 \\ &- \frac{25}{3}M_{3/2}M_{5/2} + \frac{16}{3}M_{3/2}M_{7/2} - \frac{160}{9}M_{5/2}M_{7/2}, \end{aligned} \tag{8}$$

where $M_\gamma = M(2/3;\gamma;-x^2/4)$ is the confluent hypergeometric function. The results of calculating the correlation coefficient are presented in Fig. 1.

3. In A. M. Obukhov' s theory[2] one uses the dissipation averaged over the volume of a sphere of radius R:

$$\varepsilon_R = \frac{3}{4\pi R^3}\iiint_V \varepsilon(\mathbf{r})\,d\mathbf{r}. \tag{9}$$

Fig. 1

Figure 1: Fig. 1

Starting from the formulas of the theory of locally isotropic turbulence, one can calculate the variance of such averaged dissipation as a function of the radius R of the averaging sphere. Here we shall determine this variance only in the inertial interval, although by means of formula (8) this could also be done over the entire equilibrium interval. For $R \ll \lambda_0$ this quantity is determined by formula (4).

Fig. 1

Let us first note that in the case of sufficiently rapid decay of the correlation (faster than r^{-3}), when the integral exists

$$\iiint_{-\infty}^{\infty} B_{\varepsilon\varepsilon}(\mathbf{r})\, d\mathbf{r} = 4\pi \int_0^{\infty} r^2 B_{\varepsilon\varepsilon}(r)\, dr = B_{\varepsilon\varepsilon}(0) V_0, \tag{10}$$

the variance of ε_R for sufficiently large R is asymptotically represented by the formula

$$D(\varepsilon_R) \simeq B_{\varepsilon\varepsilon}(0) \frac{V_0}{V_R},$$

i.e., independently of the specific form of the correlation function the quantity $D(\varepsilon_R)$ decreases inversely proportional to the averaging volume. If, however, the correlation function $B_{\varepsilon\varepsilon}(r)$ decays more slowly than r^{-3}, so that integral (10) diverges, then the variance of the averaged dissipation, although it tends to zero as $V_R \to \infty$, does so more slowly than R^{-3}. This is precisely the situation in the case of the correlation function (6).

It will be convenient for us to rewrite formula (9) in the form

$$\varepsilon_R = \frac{3}{4\pi R^3} \iiint_{-\infty}^{\infty} \varepsilon(\mathbf{r}) \chi(\mathbf{r})\, d\mathbf{r}; \qquad \chi(\mathbf{r}) = \begin{cases} 1, & |\mathbf{r}| \leq R, \\ 0, & |\mathbf{r}| > R. \end{cases}$$

Hence for the variance of ε_R one obtains the expression

$$D(\varepsilon_R) = \frac{9}{16\pi^2 R^6} \iiint_{-\infty}^{\infty} \iiint_{-\infty}^{\infty} B_{\varepsilon\varepsilon}(|\mathbf{r}_1 - \mathbf{r}_2|) \chi(\mathbf{r}_1) \chi(\mathbf{r}_2)\, d\mathbf{r}_1 d\mathbf{r}_2.$$

It is convenient next to pass to the spectrum of the quantity ε:

$$D(\varepsilon_R) = \frac{9}{16\pi^2 R^6} \iiint_{-\infty}^{\infty} \chi(\mathbf{r}_1)\, d\mathbf{r}_1 \iiint_{-\infty}^{\infty} \chi(\mathbf{r}_2)\, d\mathbf{r}_2 \iiint_{-\infty}^{\infty} f_\varepsilon(\mathbf{k}) e^{i\mathbf{k}(\mathbf{r}_1-\mathbf{r}_2)}\, d\mathbf{k}.$$

Carrying out here the integration over $\mathbf{r}_1$ and $\mathbf{r}_2$ and over the angular variables entering $d\mathbf{k}$, and making the substitution $y = kR$, we obtain the single integral:

$$D(\varepsilon_R) = \frac{36\pi}{R^3} \int_0^\infty f_\varepsilon\left(\frac{y}{R}\right) \left(\frac{\sin y}{y} - \cos y\right)^2 \frac{dy}{y^2}. \tag{11}$$

To find the asymptotic form of $D(\varepsilon_R)$ for large R (in the inertial range), one must know the behavior of the dissipation spectrum in the region of small wave numbers. For the three-dimensional dissipation spectrum, as for the three-dimensional spectrum of any other isotropic scalar field, the relation to the correlation function has the form

$$f_\varepsilon(k) = \frac{1}{2\pi^2} \int_0^\infty \frac{\sin kr}{kr}\, r^2 B_{\varepsilon\varepsilon}(r)\, dr.$$

With the aid of this formula and expression (6), we find the form of the spectrum for small k (in the inertial range of wave-vector space):

$$f_\varepsilon(k) = \frac{1109\sqrt{3}\,\Gamma(4/3)}{576\pi^2}\, C^2\nu^2\varepsilon^{-4/3}k^{-1/3} \simeq 0.302\, C^2\nu^2\varepsilon^{-4/3}k^{-1/3}. \tag{12}$$

Substituting this expression into (11), we obtain:

$$D(\varepsilon_R) = \frac{27 \cdot 1109\sqrt{3}\,\Gamma(4/3)\Gamma(5/3)}{320\pi \cdot 2^{2/3}}\, C^2\nu^2\varepsilon^{-4/3}R^{-8/3} \simeq 26.2\, C^2\nu^2\varepsilon^{-4/3}R^{-8/3}. \tag{13}$$

Let us emphasize that, upon adopting Millionshchikov’ s hypothesis, formula (13) for $D(\varepsilon_R)$ at $\lambda_0 \ll R \ll L$ is determined solely by the expressions for the velocity structure functions in the inertial range.

It should also be noted that the singularity (integrable) of the dissipation-field spectrum (12) near the origin of wave-number space is again associated with the insufficiently rapid decay of the dissipation correlation in the inertial range. All these results in the inertial range are qualitatively preserved even if Millionshchikov’ s hypothesis is abandoned, owing to the similarity of the turbulence regime.

In conclusion, I express my deep gratitude to A. M. Obukhov for suggesting the topic and for useful advice during the course of its execution.

Institute of Atmospheric Physics
Academy of Sciences of the USSR

Received
5 I 1962

CITED LITERATURE

1. A. N. Kolmogorov, DAN, **30**, No. 4 (1941).
2. A. M. Obukhov, *J. Fluid Mech.*, **12**, part 4 (1962).
3. A. N. Kolmogoroff, *J. Fluid Mech.*, **12**, part 4 (1962).
4. A. N. Kolmogorov, DAN, **32**, No. 1 (1941).
5. E. A. Novikov, DAN, **139**, No. 2 (1961).

Note: Figure translations are in progress. See original paper for figures.