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Abstract

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ON THE ASYMPTOTICS OF SOLUTIONS OF A FREE-BOUNDARY PROBLEM FOR THE HEAT EQUATION

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In this note we consider the asymptotic behavior, as $t \to \infty$, of the solution of the following problem. In the domain $D\{0 \le x \le s(t),\ 0 \le t \le T\}$, where $s(t)$ is an unknown function, find a solution of the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \tag{1}$$

which satisfies the following boundary conditions:

$$u\Big|_{x=0} = f_1(t), \qquad u\Big|_{x=s(t)} = f_2(t), \qquad \frac{\partial u}{\partial x}\Big|_{x=s(t)} = g(t). \tag{2}$$

A solution of such a problem is a pair of functions $u(x,t)$, $s(t)$, of which $u(x,t)$ satisfies equation (1) in the domain D, and conditions (2) are fulfilled.

Under certain restrictions on the functions $f_1(t)$, $f_2(t)$, $g(t)$, the existence and uniqueness of the solution of this problem were proved by T. D. Venttsel' ([1]). Analogous problems were considered in the seminar of O. A. Oleinik. Similar problems arise in solving filtration problems taking bound water into account (see ([2])).

Theorem. Let the functions f_1, f_2, g satisfy the conditions

$$\begin{aligned} |f_1(t) - a_1| &\le \varepsilon(t), \\ |f_2(t) - a_2| &\le \varepsilon(t), \\ |g(t) - b| &\le \varepsilon(t), \end{aligned} \tag{3}$$

where the continuous function $\varepsilon(t) \to 0$ as $t \to \infty$; a_1, a_2, b are certain constants, with $a_2 > a_1$, $b > 0$. Suppose that there exists a solution $u(x,t)$, $s(t)$ of problem (1), (2), and that

$$s(t) \le \frac{a_2 - a_1}{b}.$$

(In the case when the existence of a solution of problem (1), (2) has been proved, the function $s(t)$ satisfies this condition.)

Then

$$|u(x,t) - bx - a_1| \le M_1 \psi(t); \tag{4}$$

$$\left|\frac{a_2 - a_1}{b} - s(t)\right| \le M_2 \psi(t), \tag{5}$$

where

$$\psi(t) = ce^{-ct}\left(\int_0^t e^{cz} \sup_{\tau \ge z} |\varepsilon(\tau)|\, dz + \frac{1}{c} \sup_{\tau \ge 0} |\varepsilon(\tau)|\right);$$

M_1, M_2, c are certain positive constants.

Proof. Consider the function $v(x,t) = u(x,t) - bx - a_1$, which satisfies the heat equation. Make the substitution $v = \sin(k_1 x + k_2)\psi(t) w(x,t)$, where k_2 is an arbitrary positive constant, and

$$k_1 < \left(\frac{\pi}{2} - k_2\right)/s, \qquad s = \frac{a_2 - a_1}{b}.$$

Put

$$y(t) = \sup_{\tau \ge t} |\varepsilon(\tau)|.$$

It is clear that $y(t) \to 0$ as $t \to \infty$, and $y(t) > 0$ for all t.

Let

$$\psi(t) = ce^{-ct}\left(\int_0^t e^{cz} y(z)\, dz + \frac{y(0)}{c}\right),$$

where $c = k_1^2$.

The function $\psi(t)$ has the following properties:

1) $\psi(t) > 0$ for all t;

2) $\psi(t) \to 0$ as $t \to \infty$; indeed,

$$\lim_{t\to\infty}\left[ce^{-ct}\int_0^t e^{cz}y(z)\,dz + y(0)e^{-ct}\right] =$$

$$= \lim_{t\to\infty}\frac{ce^{ct}y(t)}{e^{ct}} + \lim_{t\to\infty} y(0)e^{-ct} = 0;$$

3) $\psi(t) > y(t)$ for all t; indeed,

$$\psi(t) = ce^{-ct}\left\{\left[\frac{1}{c}y(z)e^{cz}\right]_0^t - \frac{1}{c}\int_0^t e^{cz}\,dy + \frac{y(0)}{c}\right\} =$$

$$= ce^{-ct}\left(-\frac{y(0)}{c} + \frac{1}{c}y(t)e^{ct} - \frac{1}{c}\int_0^t e^{cz}\,dy + \frac{y(0)}{c}\right),$$

$$\frac{\psi(t)}{y(t)} = 1 - \frac{e^{-ct}}{y(t)}\int_0^t e^{cz}\,dy > 1,$$

since $y(t)$ is a nonincreasing function;

4) $\left|\frac{\psi'(t)}{\psi(t)}\right| \leqslant c$; we have

$$\frac{\psi'}{\psi} = -c\left(1 - \frac{y}{\psi}\right)$$

and, since

$$\frac{y}{\psi} < 1,$$

then

$$\left|\frac{\psi'}{\psi}\right| \leqslant c.$$

The function $w(x,t)$ satisfies the equation

$$w_{xx} + 2k_1 \operatorname{ctg}(k_1x + k_2)w_x - \left(k_1^2 + \frac{\psi'}{\psi}\right)w = w_t; \tag{6}$$

here $k_1s + k_2 < \frac{\pi}{2}$ and $\sin(k_1s + k_2) \neq 0$, $\cos(k_1s + k_2) \neq 0$. In view of the inequality

$$-k_1^2 - \frac{\psi'}{\psi} \leqslant 0,$$

the maximum principle holds for equation (6).

Let us estimate the function

$$w = \frac{v}{\sin(k_1 x + k_2)\psi(t)}$$

on the boundary of the domain D. We have

$$|w||_{x=0} = \left|\frac{f_1(t) - a_1}{\sin k_2\, \psi(t)}\right| \leqslant \frac{\varepsilon(t)}{\sin k_2\, \psi(t)} \leqslant M_3,$$

since

$$\psi(t) \geqslant |\varepsilon(t)|.$$

For $x = s(t)$,

$$w|_{x=s(t)} = \frac{v_x|_{x=s(t)}}{\psi(t) k_1 \cos(k_1 s + k_2)} - \frac{w_x|_{x=s(t)}}{k_1 \operatorname{ctg}(k_1 s + k_2)}.$$

If $\max|w(x,t)|$ is attained at $\bar{x} = s(\bar{t})$, then at the point of a positive maximum of the function $w(x,t)$ we have $w_x \geqslant 0,\ w > 0$; hence

$$w(x,t)|_{x=s(t)} \leqslant \frac{v_x|_{x=s(\bar{t})}}{\psi(\bar{t})\cos(k_1 s + k_2)};$$

the case of a negative minimum is treated analogously.

Therefore

$$|w||_{x=s(t)} \leq \left|\frac{v_x|_{x=s(\bar{t})}}{k_1 \psi(\bar{t})\cos(k_1 s + k_2)}\right| \leq \left|\frac{\varepsilon(\bar{t})}{k_1 \psi(\bar{t})\cos(k_1 s + k_2)}\right| \leq M_4.$$

Since the function $w(x,t)$ satisfies equation (6), for which the maximum principle holds, we have $|w(x,t)| \leq M_5$, where $M_5 = \max(M_3, M_4)$. Then

$$|v(x,t)| \leq \sin(k_1 s + k_2)\psi(t) M_5 \leq M_1 \psi(t), \tag{7}$$

i.e.

$$|u(x,t) - bx - a_1| \le M_1 \psi(t).$$

We shall show that the function $s(t) \to \dfrac{a_2 - a_1}{b}$ as $t \to \infty$. We have

$$v|_{x=s(t)} = f_2(t) - a_2 + a_2 - bs(t) - a_1. \tag{8}$$

From relation (7) it follows that

$$|v(x,t)||_{x=s(t)} \le M_1 \psi(t). \tag{9}$$

From (8) and (9) we obtain

$$|a_2 - a_1 - bs(t)| \le |M_1 \psi(t)| + |a_2 - f_2(t)|; \qquad \left|\frac{a_2 - a_1}{b} - s(t)\right| \le M_2 \psi(t),$$

since $|\varepsilon(t)| \le \psi(t)$.

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References

1. T. D. Venttsel, DAN, **131**, No. 5 (1960).

2. V. A. Florin, Izv. AN SSSR, OTN, No. 11, 1625 (1951).

Note: Figure translations are in progress. See original paper for figures.