MATHEMATICS

1962

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Abstract

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MATHEMATICS

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ANALYTIC FUNCTIONS IN TUBE DOMAINS

(Presented by Academician P. S. Novikov, 27 III 1962)

A **tube domain** (t.d.) D in n-dimensional complex space C^n with base $\operatorname{Im} D$, where $\operatorname{Im} D$ is a domain in n-dimensional real space R^n, is the domain consisting of points $z \in C^n$ for which $y = (\operatorname{Im} z)/i \in \operatorname{Im} D$. A t.d. is called **essentially bounded** if the convex hull of its base contains no entire straight lines. This name is connected with the fact that a t.d. is analytically equivalent to a bounded domain if and only if it is essentially bounded. Bochner' s theorem (^{1}) holds, according to which, in order that a t.d. be a domain of holomorphy, it is necessary and sufficient that its base be convex. The **type** of a t.d. D is the maximal cone* V_D whose translate can be placed in $\operatorname{Im} D$. The simplest domain having type V is the t.d. for which $\operatorname{Im} D = V$. Such domains are called radial. It is clear that for an essentially bounded t.d. D the cone V_D contains no straight lines. If $\operatorname{Im} D$ is a bounded domain, then we shall assume that $V_D = \{0\}$.

The significance of t.d.' s lies in the fact that they are the natural domains of definition of multidimensional Fourier transforms. This circumstance is a generalization of the well-known fact that the Fourier transform of functions on a line (half-line) is naturally considered in a strip (half-plane) depending on the growth of the function, or, in other words, in a one-dimensional t.d. The corresponding result is conveniently formulated in the language of normed rings (^{2}). We shall proceed in the same way in the n-dimensional case. Let V be a convex cone of dimension n in R^n, and let $a(t)$ be a continuous function on V satisfying the condition:

$$a(t_1 + t_2) \leqslant a(t_1)a(t_2).$$

Consider the ring $L_V^{(a)}$ of complex measurable functions on V, with norm

$$\|f\| = \int_V |f(t)|a(t)\,dt$$

(dt is Euclidean measure), and with convolution as multiplication:

$$f * g(t) = \int f(x)g(t - x)\,dx$$

(the integral is taken over the set of points $x \in V$ such that $t - x \in V$ [over a "segment" in the sense of the cone V]).

Theorem 1[**]**.** *The set* $\mathcal{M}(L_V^{(a)})$ *of maximal ideals of the normed ring* $L_V^{(a)}$ *is the closure of an essentially bounded t.d. of holomorphy of type* V^**, where* V^* *is the cone conjugate*[***] *to the cone* V*. Every essentially bounded t.d. of holomorphy* D *is the set of maximal ideals of a ring* $L_{V_D^*}^{(a)}$ *for some weight* $a(t)$*.*

Elements of $L_V^{(a)}$ are represented in the form of analytic functions on $\mathcal{M}(L_V^{(a)})$ by means of the usual formulas from the theory of Fourier transforms:

$$f(t) \mapsto \tilde{f}(z) = \int_V f(t) \exp(i(t,z))\, dt,$$

where

$$(t,z) = \sum t_i z_i.$$

For every of a ray k contained in the cone V and issuing from its vertex, put $\alpha(k) = \lim \ln(a(t)) \cdot (-|t|)^{-1}$ as $|t| \to \infty$, $t \in k$, where $|t| = \sum |t_i|$. Then the base of the tube domain $\mathcal{M}(L_V^{(a)})$ will be a convex domain for which $\alpha(k)$ is the support function, i.e., the maximal domain whose orthogonal projection on k is the half-line: $|t| > \alpha(k)$. It is clear that the inverse correspondence is not unique, since $\mathcal{M}(L_V^{(a)})$ depends only on $\alpha(k)$, whereas $a(t)$ and $L_V^{(a)}$ are not determined by these quantities. The simplest weight has the form $a_0(t) = m_D(t) = \exp(-\alpha(k)|t|)$ for $t \in k$. It is constructed in the following way from the domain D: $m_D(t) = \max \exp(i(t,z))$, $z \in D$. We shall call the skeleton Ω_D of the tube domain D the set of boundary points ∂D at which the maximum of the function $\exp(i(t,z))$ is attained for $t \in V_D^*$; Ω_D is a tubular set. On $\operatorname{Im}\Omega_D$ prescribe a nonnegative σ-finite measure μ (^{4}) such that all open sets in $\operatorname{Im}\Omega_D$ are μ-measurable,

$$\sup \operatorname{vrai} \exp(-t,y) = m_D(t)$$

and

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[*] We consider open cones with vertex at the origin.

[**] Theorem 1 for the case when the cone V is an octant (a direct product of half-lines), under a certain additional restriction on $a(t)$ leading to restrictions on the class of t.d.' s, was proved by B. S. Mityagin (^{3}).

[***] The cone V^* is the intersection of half-spaces, each of which is bounded by a hyperplane passing through the vertex of V orthogonally to one of its generators and contains this generator. The cone V^* is always convex; if the cone V contains no straight lines, then $\dim V^* = n$, and conversely. For $V = \{0\}$ we put $V^* = R^n$.

$$b_{\mu,D}(t) = (2\pi)^n \int_{\operatorname{Im}\Omega_D} \exp(-t, y)\,\mu(dy) < \infty$$

for all $t \in V_D^*$. Everywhere in what follows it is assumed that the measure μ has these properties. Then as the weight $a_1(t)$ one may take $\left(b_{\mu,D}(t)\right)^{1/2}$. In this case $\alpha(k)$ for $a_0(t)$ and $a_1(t)$ coincide, since for every nonnegative essentially bounded μ-integrable function f

$$\lim_{\rho \to +\infty} \left(\int_{\operatorname{Im}\Omega_D} (f(y))^\rho\,\mu(dy) \right)^{1/\rho} = \sup \operatorname{vrai} f(y). \tag{1}$$

The Fourier transform effects an isometric mapping of the Hilbert space $L^2(V_D^*, \mu)$ of functions on V_D^* with square of the modulus integrable with respect to the measure $b_{\mu,D}(t)\,dt$ onto the Hilbert space $L^2(\Omega_D, \mu)$ of functions on Ω_D, analytically continuable to D, with square of the modulus integrable on Ω_D with respect to $\mu(dy)\,dx$, where dx is Euclidean measure on $\mathbb{R}^n$. The integral expressing $\widehat{f}(z)$ in terms of $f(t)$ converges in mean for $z \in \Omega_D$ (in the metric $L^2(V_D^*, \mu)$) and uniformly in every tube domain $D_1 \Subset D^*$.

A Cauchy–Szegő kernel for an essentially bounded tube domain D of holomorphy relative to the measure μ on $\operatorname{Im}\Omega_D$ is a function $S(w, z)$, analytic in w for $z \in \Omega_D \cup D$, antianalytic in z for $w \in D$, and such that

$$\int_{\Omega_D} S(w, z) f(z)\,\mu(dy)\,dx = f(w), \quad w \in D$$

for $f \in L^2(\Omega_D, \mu)$.

Theorem 2. *The Cauchy–Szegő kernel for the domain D relative to a fixed measure μ exists and is unique. It has the form*

$$S(w, z) = \int_{V_D^*} \exp\bigl(i(t, w - \bar{z})\bigr)\, b_{\mu,D}^{-1}(t)\,dt, \tag{2}$$

where the integral converges uniformly if $w \in D_1$, $z \in \Omega_D$ for $D_1 \Subset D$.

Formula (2) for radial tube domains was obtained by Bochner ([5]). It is comparatively rare that one can find the kernel $S(w, z)$ in explicit form. In ([6],[7]) the Cauchy–Szegő kernels are computed for all homogeneous tube domains. A generalized Cauchy–Szegő kernel for a tube domain D relative to the measure μ is a kernel $\check{S}(w, z)$, where $w \in D$, $z \in \Omega_D$, possessing all the properties of the kernel S except antianalyticity in z. The generalized Cauchy–Szegő kernel is no longer unique and, owing to the arbitrariness in its choice, such a kernel can be written in a unified form for whole series of domains. We indicate one method of obtaining kernels $\check{S}(w,z)$.

Theorem 3. *Let* $\varphi_1(y),\dots,\varphi_n(y)$ *be real functions on* $\operatorname{Im}\Omega_D$, *nonzero almost everywhere (with respect to* μ*), such that*

$$b_{\mu,D,\varphi}(t) = (2\pi)^n \int_{\operatorname{Im}\Omega_D} \exp(-2t,\varphi(y))\,\mu(dy) < \infty, \quad \text{where} \ \ \varphi(y) = \{\varphi_i(y)\},$$

$t \in V_D^*$. Then the function

$$\check{S}(w,z) = \int_{V_D^*} \exp(it, w - z + 2i\varphi(y))\, b^{-1}_{\mu,D,\varphi}(t)\, dt \tag{3}$$

is the generalized Cauchy–Szegő kernel if and only if the integral in (3) converges uniformly for $w \in D_1,\ z \in \Omega_D$ for every domain $D_1 \Subset D$.

The conditions that must be imposed on $\varphi_i(y)$ are obtained explicitly from the following assertion.

Proposition 1. For the absolute convergence in D of the integral

$$\int_{V_D^*} f(t)\exp(it,z)\,dt,$$

where $f(t)$ is a locally summable function, it is sufficient that

$$\overline{\lim}\ \ln(|f(t)|\, m_D(t))/|t|^{-1} \le 0 \quad \text{as } |t| \to \infty.$$

Proposition 2. The function represented by the integral (3) is the generalized Cauchy–Szegő kernel if and only if on Ω_D^*

$$\sup \operatorname{vrai}(t, -2\varphi(y) + y) = \sup \operatorname{vrai}(t, -2\varphi(y)) - \ln m_D(t), \quad t \in V_D^*. \tag{4}$$

If, for some tube domain D_0, the Cauchy–Szegő kernel S_0 has been computed with respect to the measure μ_0, and for another domain D of the same type it has been possible to choose the functions φ_i and the measure μ so that $b^*_{\mu_0,D_0}(t) = b_{\mu,D,\varphi}(t)$ and condition (4) is satisfied, then the kernel S_0 generates

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[*] We use the notation $D_1 \Subset D$ if the boundaries ∂D_1 and ∂D are everywhere at positive distance, and $D_1 \Subset\subset D$ if the domain D_1 lies strictly inside D (i.e., ∂D_1 and ∂D have no common points even at infinity).

a generalized kernel $\check{S}$ for D. For simplicity we formulate the result for tube domains in C^2 with smooth boundary.

Proposition 3. Let D_0 be a domain in C^2 whose base is given by the condition $y_2 > \chi_0(y_1)$, where χ_0 is a twice differentiable function, with $\chi_0''(y_1) > 0$, and let the Cauchy–Szegő kernel for D_0 with respect to a certain measure μ_0 have the form $S_0(w,z) = s_0((w-\bar{z})/i)$. Let now D be any domain in C^2 having the same type as D_0, and let $\operatorname{Im} D$ be given by the condition $y_2 > \chi(y_1)$, where χ is also a twice differentiable function. Define, from the functional equation $\chi_0'(\varphi_1(y_1)) = \chi'(y_1)$, the function $\varphi_1(y_1)$, and put $\varphi_2(y_1) = \chi_0(\varphi_1(y_1))$. If, moreover, $\chi_0''(\varphi_1(y_1)) < 2\chi''(y_1)$, then

$$\check{S}(w,z) = s_0((w-z)/i + 2\varphi(y))$$

is the generalized Cauchy–Szegő kernel for D with respect to the measure

$$\mu(dy_1) = \mu_0(d(\varphi_1(y_1))).$$

Now let us give examples.

1. Let $\operatorname{Im} D_0$ be given by the condition $y_2 > 2y_1^2$ and let $\mu_0(dy_1) = dy_1$. The cone V_D^* is a vertical half-plane. Put $\eta_j = (w_j - z_j)/i$. Then

$$S_0(w,z) = s_0(\eta_1,\eta_2) = \pi^{-2}(\eta_1 - \eta_2^2)^{-2}.$$

This kernel extends to domains for which $\chi''(y_1) > 2$. In this case $\varphi_1(y_1) = \chi'(y_1)/4$.

2. Let now $\chi_0(y_1) = 1/y_1$, $y_1 > 0$. The type of this domain is the first quadrant. Put $\mu(dy_1) = (y_1)^{-1/2}\,dy_1$. Then

$$s_0(\eta_1,\eta_2) = 2^{-5/2}\pi^{-2}(\eta_1)^{-1/2}(\sqrt{\eta_1\eta_2} - 2)^{-2}.$$

This kernel extends to domains for which

$$\chi''(y_1) > (-\chi'(y_1))^{3/2},$$

and $\varphi_1(y_1) = (-\chi'(y_1))^{-1/2}$.

The results given on generalized Cauchy–Szegő kernels are analogous to the results of L. A. Aizenberg (8) for polycircular domains.

Let $A(D)$ be the space of functions analytic in the tube domain D, with the topology of uniform convergence on each compact set $K \Subset D$. We shall now describe a certain space $T'(V_D^*)$ of generalized functions on V_D^*, dual to the space $A(D)$ with respect to the Fourier transform. We shall restrict ourselves to the case when the cone V_D^* contains no straight lines. The space of test functions $T(V_D^*)$ consists of functions $\psi(t)$ on V_D^* that extend to the whole space C^n as entire functions $\psi(w)$ of exponential type, and such that for every $\xi \in \operatorname{Im}\Omega_D$ there exists a vector-function $H_\xi(\theta) \in R^n$, $\theta = \{\theta_1, \dots, \theta_n\}$, $0 \le \theta_i \le 2\pi$, such that $(\xi - H_\xi(0)) \in V_D$

and $\psi(t\exp(i\theta)) \le C_\xi \exp(H_\xi(\theta), t)$ for all $t \in V_D^*$. The sequence $\psi_\nu \to 0$ in $T(V_D^*)$ if, for all ψ_ν, the estimate indicated above holds with common C_ξ and $H_\xi(\theta)$, and, moreover, ψ_ν converge uniformly to zero in every bounded domain.

Let us show how one can establish a correspondence between the spaces $A(D)$ and $T'(V_D^*)$. Fix on $\operatorname{Im}\Omega_D$ some measure μ, from which we require that $b_{\mu,D}(t)$ extend to $\mathbb{C}^n$ as an entire function (such a measure always exists). Then

$$e_z(t) = \exp(i(t,z))\, b_{\mu,D}^{-1}(t) \in T(V_D^*)$$

for $z \in D$ by virtue of (1). To a function $f(z) \in A(D)$ we associate the functional from $T'(V_D^*)$ equal to $f(z)$ on $e_z(t)$; by this condition it is completely determined. The elements of the space $A(\overline{D})$, dual to $T(V_D^*)$, are functions analytic in some domain $D_1 \supset\supset D$ and belonging to $L^2(\Omega_D, \mu)$ (independently of μ). To the topology described above in $T(V_D^*)$ there corresponds in $A(\overline{D})$ the topology of the inductive limit over domains $D_1 \supset\supset D$. For simplicity we shall give the general form of a linear functional in $A(D)$ ($A(\overline{D})$) only for the case when V_D is an octant. Let $\xi \in \operatorname{Im}\Omega_D$. Consider a radial tube domain whose base is the octant with vertex at ξ, and some bounded polycylinder contained in this domain (the direct product of one-dimensional bounded domains) D_ξ with base Ω_ξ. Choose D_ξ for each $\xi \in \operatorname{Im}\Omega_D$ so that $D_{\xi_1} \not\subset D_{\xi_2}$, $\Omega_{\xi_1} \cap \Omega_{\xi_2} = \emptyset$ for $\xi_1 \neq \xi_2$, and the boundaries ∂D_ξ form a continuous hypersurface bounding the domain $D_0 = \bigcup D_\xi$. Then $D_0 \subset D$, and $\Omega_0 = \bigcup \Omega_\xi$ is the base of the domain D_0. Every functional on $A(D)$ can be represented in the form

$$\int_{\Omega_D} f(z)\overline{\varphi(z)}\, \mu(dy)\, dx,$$

where $f \in A(D)$, $\varphi \in A(\overline{D})$, and the integral is summed in the sense of Abel, the domain D being approximated by domains D_0 of the kind just indicated. Every functional can also be represented in the form

$$\int_{\operatorname{Im}\Omega_D} \mu(d\xi) \int_{\Omega_\xi} f(z)\varphi(2i\xi - z)\, dz,$$

where the Ω_ξ correspond to some domain D_0 constructed above, such that the function $\varphi(2i\xi - z)$ is defined for $z \in \Omega_\xi$. Under this condition the integral does not depend on the choice of D_0. Let us note that if we now consider the space $B(\Omega_D)$ of boundary values on Ω_D of functions from $A(\overline{D})$ with the induced topology, then the conjugate space $B'(\Omega_D)$ can be interpreted as the space of generalized boundary values of functions from $A(D)$. From these values one can reconstruct functions from $A(D)$ by means of the Cauchy–Szegő kernels, interpreting the formulas as is customary in the theory of generalized functions. All the facts indicated can be generalized to the case of arbitrary cones. In constructing the spaces T we in fact made no essential use of the boundedness of the domain D and the convexity of its base. On the other hand, it is clear that $T(V_D^*)$ depends only on the convex hull of $\operatorname{Im} D$. Hence the Bochner theorem (1) formulated at the beginning of the article follows. Further, one can directly establish an isomorphism of the space $T(V_D^*)$ for essentially bounded domains of the same dimension. It follows that the linear topological spaces $A(D)$ are isomorphic for all essentially bounded tube domains of one and the same dimension.

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Received
20 III 1962

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