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Physics

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1962

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Abstract

Full Text

Physics

Yu. A. RYLOV

ON THE RELATIVE LOCALIZATION OF THE GRAVITATIONAL FIELD

(Presented by Academician N. N. Bogolyubov on 6 II 1962)

According to Einstein's theory of gravitation, gravitational forces manifest themselves in the curvature, and only in the curvature, of space-time; therefore, if we consider a region of space-time that is small compared with the radius of curvature, we may neglect gravitation and regard the region as flat. This is precisely what is expressed by the equivalence principle, which asserts that, by a suitable consideration, the gravitational field can be eliminated at any individual point of space-time. This is otherwise interpreted as the impossibility of localizing the gravitational field. Thus, from the standpoint of the equivalence principle it is meaningless to ask whether there is a gravitational field at a given point regardless of anything else. However, the question of the magnitude of the gravitational field at a point x relative to the field at a point x' has a definite meaning. Let, for example, the gravitational field at the point x' be zero. This condition determines the gravitational field at the point x, although not completely. It turns out that it is possible to indicate such an invariant procedure that the condition that the gravitational field vanish at the point x' completely determines the gravitational field at the point x.

Such a description of the gravitational field will necessarily be two-point, but it makes it possible, in a certain sense, to localize the gravitational field at all points relative to an arbitrary point x' (which we shall call the reference point), where there is no field. A valuable property of such localization is that it does not contradict the equivalence principle.

Let there be in space-time (which we denote by V_4) some coordinate system K. Let x' be an arbitrary point of V_4. Consider the four-dimensional Euclidean space $E_{x'}$, tangent to V_4 at the point x'. Map V_4 onto $E_{x'}$ in such a way that the geodesics passing through x' in V_4 are mapped into straight lines in $E_{x'}$, the angles between geodesics at the point x' remain unchanged under the mapping, and the intervals from an arbitrary point M in V_4 and its image M^* in $E_{x'}$ to the point x', measured along the geodesics respectively in V_4 and $E_{x'}$, are equal. Such a mapping is one-to-one in the region where the geodesics issuing from x' do not intersect. Under this mapping the coordinate system K in V_4 is mapped into a coordinate system $K_{x'}$ in $E_{x'}$. Now the coordinates x^α number both the points of the space V_4 and the points of the space $E_{x'}$. Let $g_{\mu\nu}(x)$, $\gamma^\alpha_{\beta\gamma}(x)$,

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and $G_{\mu\nu}(x,x')$, $\Gamma^{\alpha}_{\beta\gamma}(\overline{x,x'})$ be the metric tensor and the Christoffel symbols, respectively, in V_4 in the coordinate system K, and in $E_{x'}$ in the coordinate system $K_{x'}$. In general, we shall denote two-point quantities by capital letters, and one-point quantities by lowercase letters. The tensor

$$Q^{\alpha}_{\beta\gamma} = \gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\beta\gamma} \tag{1}$$

will describe the gravitational field at the point x relative to the point x'. The condition $Q^{\alpha}_{\beta\gamma} = 0$ is the necessary and sufficient condition for the Euclidean character of the space V_4. The quantity $Q^{\alpha}_{\beta\gamma}$, in contrast to the Christoffel symbols, is ···

is a tensor; furthermore,

$$[Q^{\alpha}_{\beta\gamma}] = 0. \tag{2}$$

Here and below the square brackets mean that $x = x'$ has been set. Condition (2) is an invariant formulation of the equivalence principle. Indeed, it is always possible to make the gravitational field $Q^{\alpha}_{\beta\gamma}$ vanish at an arbitrary point x; for this it is sufficient to choose the reference point $x' = x$.

Thus, the compatibility of the equivalence principle with the idea of passing from a Riemannian space to a Euclidean one is achieved here by introducing a continuum of Euclidean spaces $E_{x'}$, depending on the coordinates of the reference point.

Since in what follows we shall have to operate with two-point quantities, in particular with two-point tensors (briefly, bitensors), we agree that indices without primes refer to the point x, while indices with primes refer to the point x'. Further, when this does not lead to misunderstandings, we shall omit the argument, bearing in mind that the presence or absence of a prime on indices indicates the argument; for example: $g^{\alpha'\beta'}$ is $g^{\alpha'\beta'}(x')$, $\gamma^{\alpha}_{\beta\gamma}$ is $\gamma^{\alpha}_{\beta\gamma}(x)$. Ordinary derivatives will be denoted by the symbol ∂, or by a comma before the corresponding index; covariant derivatives with the Christoffel symbols $\gamma^{\alpha}_{\beta\gamma}$ or $\gamma^{\alpha'}_{\beta'\gamma'}$ by the sign ∇, or by a vertical stroke before the corresponding index. Covariant derivatives in the tangent space $E_{x'}$ with the Christoffel symbol $\Gamma^{\alpha}_{\beta\gamma}$ will be denoted by the symbol $\widetilde{\nabla}$, or by two vertical strokes before the corresponding index. The presence or absence of a prime on the index of a derivative indicates that the derivative is taken respectively with respect to x' or to x.

Let us write the action in the form

$$S(x') = \int_{\Omega} L(x,x')\sqrt{-g}\, d^4x; \tag{3}$$

$$L(x,x') = L_m(x) + \frac{1}{2\varkappa} L_g(x,x'), \tag{4}$$

where L_m is the Lagrangian of matter, $\varkappa$ is Einstein' s gravitational constant, and L_g is the Lagrangian of the gravitational field, taken in the form

$$L_g = L_g(x, x') = g^{\mu\beta}(Q^{\alpha}_{\beta\gamma} Q^{\gamma}_{\alpha\mu} - Q^{\alpha}_{\mu\beta} Q^{\gamma}_{\alpha\gamma}), \tag{5}$$

where $Q^{\alpha}_{\beta\gamma}$ is given by (1), and also by the expression

$$Q^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(g_{\delta\beta\|\gamma} + g_{\delta\gamma\|\beta} - g_{\beta\gamma\|\delta}). \tag{6}$$

From (3), with the aid of the variational principle, we obtain the equations of motion of matter and Einstein' s equations of gravitation; here the essential point is that $\delta\Gamma^{\alpha}_{\beta\gamma}$ reduces to the variation caused by coordinate transformations, since $\Gamma^{\alpha}_{\beta\gamma}$ is the Christoffel symbol for flat space.

Let us pass in (3) to integration over flat space. For this purpose we write (3) in the form

$$S(x') = \int_{\Omega} L\Lambda^{-1}\sqrt{-D_x}\, d^4x, \tag{7}$$

$$D_x = \det\|G_{\alpha\beta}\|, \qquad \Lambda = \sqrt{D_x g^{-1}(x)},$$

where L and Λ are scalars. From the invariance of (7) with respect–

but, by virtue of Noether' s theorem, for displacements $E_{x'}$ we obtain

$$\Theta^{\alpha}_{\beta\|\alpha} = 0; \tag{8}$$

$$\Lambda\Theta^{\alpha}_{\beta} = -\sum_i \frac{\partial L_m}{\partial u_{i\|\alpha}} u_{i\|\beta} - \frac{\partial L_m}{\partial g_{\gamma\delta\|\alpha}} g_{\gamma\delta\|\beta} - \frac{1}{2\chi} \frac{\partial L_g}{\partial g_{\gamma\delta\|\alpha}} g_{\gamma\delta\|\beta} + \delta^{\alpha}_{\beta} L, \tag{9}$$

where u_i are the variables describing matter. Θ^{α}_{β} is the energy-momentum tensor of the system relative to the point x'. In contrast to the expressions obtained by other authors ([2–5]), this is a true tensor, which, in the case of flat space, goes over into the ordinary canonical energy-momentum tensor. Let us introduce the tensor of parallel displacement $P^{\gamma}_{\beta'}$. In $E_{x'}$:

$$P^{\gamma}_{\beta'} = G^{\gamma\sigma} G_{\beta'\sigma} \equiv -G^{\sigma\gamma} \frac{\partial^2 G}{\partial x^{\beta} \partial x^{\sigma}}, \tag{10}$$

where $G = G(x, x')$ is Synge' s world function ([6]). $P^{\gamma}_{\beta'}$ has the following properties:

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$$P^{\gamma}_{\beta'\|\lambda} = 0, \qquad [P^{\gamma}_{\beta'}] = \delta^{\gamma}_{\beta}. \tag{11}$$

Carrying the lower index in (9) by means of $P^{\gamma}_{\beta'}$, we obtain

$$\Theta^{\alpha}_{\beta'\|\alpha} = \frac{1}{\sqrt{-D_x}} \frac{\partial}{\partial x^{\alpha}} \left(\sqrt{-D_x}\, \Theta^{\alpha}_{\beta'} \right) = 0, \tag{12}$$

where

$$\Theta^{\alpha}_{\beta'} = P^{\gamma}_{\beta'} \Theta^{\alpha}_{\gamma}. \tag{13}$$

Integrating (12) over an arbitrary region Ω of the space $E_{x'}$, we obtain, by Gauss' s theorem,

$$\int_{\Omega} \frac{\partial}{\partial x^{\alpha}} \left(\sqrt{-D_x}\, \Theta^{\alpha}_{\beta'} \right) d^4x = \oint_{\Sigma} \Theta^{\alpha}_{\beta'} \sqrt{-D_x}\, dS_{\alpha} = \oint_{\Sigma} \Lambda \Theta^{\alpha}_{\beta'} \sqrt{-g}\, dS_{\alpha}, \tag{14}$$

where Σ is the hypersurface bounding the 4-volume Ω, and dS_{α} is an element of this hypersurface. If $\Theta^{\alpha}_{\beta'}$ vanishes at spatial infinity, then, as follows from (14), the quantity

$$P_{\beta'} = P_{\beta'}(x') = \int_{\Sigma}^{\infty} \Theta^{\alpha}_{\beta'} \sqrt{-D_x}\, dS_{\alpha}, \tag{15}$$

where Σ^{∞} is an infinite spacelike hypersurface, does not depend on Σ and is a vector at the point x'. In the case of flat space-time and Galilean coordinates in it, $P_{\beta'}$ goes over into the ordinary 4-momentum. This gives grounds for interpreting $P_{\beta'}$ as the energy-momentum vector for matter and the gravitational field relative to the point x'. For the gravitational part of the energy-momentum tensor, after straightforward calculations we obtain

$$\Lambda \Theta^{\alpha}_{g\beta} = -\frac{1}{2\chi} \left(\frac{\partial L_g}{\partial g_{\gamma\delta\|\alpha}}\, g_{\gamma\delta\|\beta} - \delta^{\alpha}_{\beta} L_g \right) =$$

$$= -\frac{1}{2\chi} \left\{ g^{\rho\sigma} \left(g^{\mu\nu} g^{\alpha\delta} - g^{\mu\delta} g^{\alpha\nu} \right) \left(Q_{\sigma\nu\beta} Q_{\delta\mu\rho} + Q_{\nu\sigma\beta} Q_{\delta\mu\rho} + Q_{\sigma\rho\beta} Q_{\delta\mu\nu} \right) - \delta^{\alpha}_{\beta} L_g \right\}, \tag{16}$$

where $Q_{\alpha\beta\gamma} = g_{\alpha\delta} Q^{\delta}_{\beta\gamma}$. The energy-momentum tensor (16) is a true tensor and, in a coordinate system that is Galilean in $E_{x'}$, is numerically equal to Einstein' s pseudotensor.

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For a static centrally symmetric field in a coordinate system where the line element has the form

$$ds^2 = e^{\nu} dt^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) - e^{\lambda} dr^2, \tag{17}$$

$\nu = \nu(r)$, $\lambda = \lambda(r)$, and the speed of light $c = 1$, the energy-momentum vector (15), calculated with respect to the point $x' = 0$ $(t' = 0,\ r' = 0)$, has the form

$$P_{0'} = E = \frac{4\pi\alpha}{\chi} e^{\nu(0)} = m e^{\nu(0)} = 4\pi e^{\nu(0)} \int_0^\infty t_0^0 r^2\, dr, \tag{18}$$

where $\alpha = \chi \int_0^\infty t_0^0 r^2\, dr$ is the gravitational radius of the system, t_0^0 is the time component of the energy-momentum tensor of the matter entering the right-hand side of Einstein' s equations. The spatial components $P_{\beta'}$ are equal to zero ($P_{i'} = 0$). For the case in which α is much smaller than the radius of the region occupied by matter, this result agrees with Møller' s result ([4]), differing from the latter only in that $P_{\beta'}$ is a vector.

Thus, even relative localization of the gravitational field places the gravitational field on an equal footing with other fields, making it possible to introduce conserved relative quantities: energy, momentum, and angular momentum. The latter is readily obtained by means of Noether' s theorem from the invariance of (7) with respect to rotations E_x.

In conclusion, I consider it my duty to express my gratitude to Prof. Ya. P. Terletskii for his attention and interest in my work, and to A. N. Gordeev for valuable discussions.

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Received
25 I 1962

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Note: Figure translations are in progress. See original paper for figures.

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