MATHEMATICS

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Abstract

Full Text

MATHEMATICS

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IMPOSSIBILITY OF AN ISOMETRIC IMMERSION IN THREE-DIMENSIONAL EUCLIDEAN SPACE OF CERTAIN MANIFOLDS WITH NEGATIVE GAUSSIAN CURVATURE

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1°. Let Λ be a two-dimensional manifold with Gaussian metric ds^2. Suppose that Λ is simply connected, complete relative to its metric, and has everywhere negative Gaussian curvature K ($K < 0$). For definiteness (without loss of generality) one may assume that Λ is the ordinary Cartesian plane (x, y) and

$$ds^2 = dx^2 + B^2\, dy^2, \tag{1}$$

where $B = B(x, y)$ is a positive function defined for all values of x, y. Since the Gaussian curvature K is given by the equality $B''_{xx} + KB = 0$, it follows that $B''_{xx} > 0$. We shall assume that the function $B = B(x, y)$ belongs to the regularity class $C^{(3)}$.

We investigate the question of the possibility of an isometric immersion of the entire manifold Λ in three-dimensional Euclidean space E_3 in the form of a locally regular surface of class $C^{(3)}$ (self-intersections of the surface are not excluded). Below a connection is established between this question and an estimate of the growth of the quantity $1/\sqrt{|K|}$.

2°. Suppose that on Λ the inequality

$$\left|\operatorname{grad} \frac{1}{\sqrt{|K|}}\right| \leq q = \text{const.} \tag{2}$$

holds.

Theorem 1. *There exists a positive number q_0 such that every manifold Λ satisfying (2), where $q < q_0$, does not admit a regular isometric immersion in E_3. For example, $q_0 = \sqrt{2}/3$.*

Theorem 1 is equivalent to the assertion that the system of equations

$$(Bl)'_y - (Bm)'_x = B'_x m, \qquad n'_x - m'_y = BB'_x l, \qquad ln - m^2 = K, \qquad (*)$$

where $K < 0$ and $\left|\operatorname{grad} 1/\sqrt{|K|}\right| \le q < q_0$ (the gradient is taken in the metric (1)), admits no solution l, m, n regular on the entire plane (x, y). We note that the system $(*)$ reduces to a hyperbolic (for $K < 0$) system of two quasilinear equations with two unknown functions.

Theorem 2. *In E_3, on every complete regular surface of negative curvature,*

$$\sup \left|\operatorname{grad} \frac{1}{\sqrt{|K|}}\right| \ge q_0.$$

The main stages of the proof of Theorem 1 are given in item **3°**. Theorem 2 is easily derived from Theorem 1 (it is enough to take into account that the universal covering of any complete surface of negative curvature is a complete simply connected manifold).

Remark. 1) Whether the set of numbers q for which estimate (2) guarantees non-immersibility of Λ is bounded is not known to the author. 2) Theorem 1 generalizes the well-known theorem of Hilbert on the impossibility of a regular immersion in E_3 of the Lobachevsky plane (which corresponds to $q = 0$). 3) Theorem 1 strengthens the results of notes ([1,3]) in the sense that it does not require

estimates of the second derivatives of the Gaussian curvature and does not require the Gaussian curvature to be separated from zero by a negative constant. However, Theorem 1 does not generalize the results of notes ([1,3]), since these results express properties of an ε-strip of a surface of negative curvature along an asymptotic line.

3°. The main stages of the proof of Theorem 1.

1) Put $k = \sqrt{|K|}$; $k > 0$. Consider in the (x, y)-plane the metric

$$dl^2 = k^2(dx^2 + B^2\, dy^2). \qquad (3)$$

We shall denote the length of a curve in this metric by l. From estimate (2) it follows (for any q) that every line which has infinite length in metric (1) also has infinite length in metric (3).

2) Denote by Ω the absolute value of the integral curvature of an arbitrary domain D of the manifold Λ (at the same time Ω is the area of the domain D in metric (3)). From estimate (2) it follows that there exist compact domains with arbitrarily large value of Ω.

3) Suppose that Λ is immersed in E_3 as a surface F. Since $K < 0$, a locally regular net of asymptotic lines is defined on F. Introduce, in a

neighborhood of an arbitrary point of F, local coordinates u, v, taking the asymptotic lines as coordinate lines. Let, in these coordinates,

$$ds^2 = e^2 du^2 + 2eg\cos\omega\, du\, dv + g^2 dv^2.$$

Then the equations hold:

$$\frac{\partial \ln(ek)}{\partial s_2} = \sin\omega\,\frac{\partial Q}{\partial s_1^*}, \qquad \frac{\partial \ln(gk)}{\partial s_1} = -\sin\omega\,\frac{\partial Q}{\partial s_2^*},$$

where $Q = \frac{1}{2}\ln k$ (see ([1],[2])). Hence

$$\frac{\partial(ek)}{\partial v} = \lambda\,(ek)(gk)\sin\omega, \qquad \frac{\partial(gk)}{\partial u} = \mu\,(ek)(gk)\sin\omega, \tag{4}$$

$$|\lambda| \le q/2, \qquad |\mu| \le q/2.$$

4) Let D be an asymptotic quadrilateral; l_1, l_2, l_3, l_4 its sides, measured in metric (3) and numbered in cyclic order. The estimates hold:

$$(l_3 + l_4) - (l_1 + l_2) \le q\Omega(D) \le \frac{q^2}{2}(l_1 + l_2 + l_3 + l_4) + qC, \tag{5}$$

$$C = \text{const}.$$

The first part of estimates (5) can be obtained from equations (4); the second part is derived from the Gauss–Bonnet formula, taking into account formulas (8a, b) of note ([2]). From (5) and from the first point of our arguments it follows: if $q < \sqrt{2}$, then the net of asymptotic lines on the whole surface F is homeomorphic to a Cartesian net on the plane. This last conclusion and estimates (5) are essentially contained in a note of P. Rozenhorn ([4]); we apply the results of note ([4]) in a somewhat modified form (in metric (3)), since we do not have the condition $k \ge 1$.

5) Assuming $q < \sqrt{2}$, we can introduce on the entire surface F one coordinate system u, v so that the coordinate net consists of complete asymptotic lines. In this case the pairs of coordinates (u, v), $-\infty < u < +\infty$, $-\infty < v < +\infty$, will correspond one-to-one to the points of Λ. Put, for brevity, $ek = \tilde{e}$, $gk = \tilde{g}$. By means of a parametrization we shall achieve the equalities $\tilde{e}(u, 0) = 1$, $\tilde{g}(0, v) = 1$. We shall regard (u, v) as Cartesian coordinates on an auxiliary Cartesian plane. We agree, however, to use only nonnegative coordinates u, v,

replacing the signs by an indication of the number of the coordinate quadrant. From the origin of the coordinates on all four semi-axes lay off segments of magnitude a ($a > 0$); through the endpoints of the segments laid off draw straight lines parallel to the coordinate axes. Thus one obtains a Descartes square T, which depicts an asymptotic quadrilateral of the surface F. In what follows certain (positive) quantities will be introduced, the notation for which is marked with the first number; these quantities refer to the part of the square T lying in the first quadrant. Analogous (positive) quantities referring to the remaining parts of the square T are denoted similarly and marked with the numbers $2, 3, 4$, corresponding to the quadrants. Quantities whose notation is not marked by numbers refer to the entire square T.

6) We have:

$$\Omega_1(a) = \Omega(T_1) = \int_0^a du \int_0^a \tilde{e}(u,v)\, \tilde{g}(u,v) \sin\omega(u,v)\, dv.$$

Introduce the quantity

$$X_1(a) = \frac{q}{2}\int_0^a \tilde{e}(u,a) \sin\omega(u,a)\, du + \frac{q}{2}\int_0^a \tilde{g}(a,v) \sin\omega(a,v)\, dv;$$

respectively,

$$\Omega(a) = \Omega_1(a) + \Omega_2(a) + \Omega_3(a) + \Omega_4(a),$$
$$X(a) = X_1(a) + X_2(a) + X_3(a) + X_4(a).$$

From the Gauss-Bonnet formula and from formulas (8 a, b) of note [(2)] it follows that

$$X(a) \geqslant \Omega(a) - C. \tag{6}$$

7) We have:

$$\Omega_1'(a) = Y_1(a) + Z_1(a),$$

where

$$Y_1(a) = \int_0^a \tilde{e}(a,v)\, \tilde{g}(a,v) \sin\omega(a,v)\, dv,$$

$$Z_1(a) = \int_0^a \tilde{e}(u,a)\, \tilde{g}(u,a) \sin\omega(u,a)\, du.$$

Accordingly,

$$\Omega'(a) = \sum \{Y_k(a) + Z_k(a)\}, \qquad k = 1, 2, 3, 4. \tag{7}$$

8) From equations (4) it follows for the first coordinate quadrant:

$$\tilde{e}(a, v) \geqslant 1 - \frac{q}{2} Y_1(a), \qquad 0 \leqslant v \leqslant a; \tag{8}$$

$$\tilde{g}(u, a) \geqslant 1 - \frac{q}{2} Z_1(a), \qquad 0 \leqslant u \leqslant a. \tag{9}$$

Analogous inequalities hold in the remaining quadrants.

9) Denote by $E(\varepsilon)$ the set of points of the numerical semi-axis $0 \leqslant a < +\infty$, where

$$\Omega'(a) < \frac{2}{q}(1 - \varepsilon), \quad \varepsilon > 0.$$

From equality (7) it follows that, for points $a \in E(\varepsilon)$,

$$Y_k(a) < \frac{2}{q}(1 - \varepsilon), \qquad Z_k(a) < \frac{2}{q}(1 - \varepsilon).$$

Hence, from inequalities (8), (9), we obtain

$$\tilde{e}(a, v) > \varepsilon \qquad (0 \leqslant v \leqslant a),$$
$$\tilde{g}(u, a) > \varepsilon \qquad (0 \leqslant u \leqslant a),$$

if $a \in E(\varepsilon)$.

10) Let $\varphi(b)$ be the measure of the intersection of $E(\varepsilon)$ with the interval $[0, b]$. We shall call the measure of the set $E(\varepsilon)$ on the half-axis $0 \leq a < +\infty$ the upper limit of the ratio of $\varphi(b)$ to b:

$$\operatorname{mes} E(\varepsilon) = \overline{\lim}_{b \to +\infty} \frac{\varphi(b)}{b}.$$

Then $\operatorname{mes} E(\varepsilon) = 0$ for every $\varepsilon > 0$ (if the contrary were allowed, then from 2) and 9) and from inequality (6) it would follow that there exists a sequence $b_n \to +\infty$ on which $\Omega(b_n)$ has exponential growth; but for $q < \sqrt{2}$ this contradicts the estimates (5)).

11) Let $\psi(b)$ be the measure of the complement of $E(\varepsilon)$ on the interval $[0, b]$. From the preceding point it follows that

$$\frac{\psi(b)}{b} \to 1 \quad \text{as } b \to +\infty.$$

Moreover,

$$\Omega'(a) \geq \frac{2}{q}(1-\varepsilon),$$

if a belongs to the complement of $E(\varepsilon)$ on $[0, +\infty)$. Hence

$$\Omega(b) \geq \frac{2}{q}(1-\varepsilon)\psi(b) \geq \frac{2}{q}(1-\varepsilon)(1-\varepsilon_1)b, \tag{10}$$

where ε_1 is any positive number and b is a sufficiently large positive number. From the estimates (5) and (10) we obtain

$$\frac{2}{q}(1-\varepsilon)(1-\varepsilon_1)b \leq \frac{8q}{2-q^2}\,b + A.$$

Here A is a constant (depending on q). Passing to the limit as $\varepsilon \to 0$, $\varepsilon_1 \to 0$, $b \to +\infty$, we find $q \geq \sqrt{2}/3$. Consequently, for $q < \sqrt{2}/3$ the manifold Λ does not admit a regular isometric immersion in E.

The theorem is proved.

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* **Correction.** In our note (1) a misprint was made in inequalities (5), where the quantity $\sqrt{E}$ should be replaced by the quantity $k\sqrt{E}$. For the further correction of the proof it is enough to assume that the lengths of the asymptotic curves of the first family are measured in the metric $dl^2 = k^2 ds^2$. The final result remains valid, since under the conditions of note (1), i.e. for $k \geq 1$, one has $dl \geq ds$.

Note: Figure translations are in progress. See original paper for figures.