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**Abstract**

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### MATHEMATICS

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# TRANSFORMATION OF ARBITRARY CANONICAL CALCULI INTO CANONICAL CALCULI OF SPECIAL TYPES

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1. Let $H$ be an arbitrary alphabet. Every canonical calculus in the alphabet $H$ (see ([1])) is given by a nonempty finite list of $H$-words,* called the **axioms** of the given calculus, and by a finite set of rules–**production schemes**, each of which has the form

$$\frac{\begin{array}{c} G_{1,1}p_{1,1}G_{1,2}p_{1,2}\dots G_{1,m_1}p_{1,m_1}G_{1,m_1+1} \\ G_{2,1}p_{2,1}G_{2,2}p_{2,2}\dots G_{2,m_2}p_{2,m_2}G_{2,m_2+1} \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ G_{n,1}p_{n,1}G_{n,2}p_{n,2}\dots G_{n,m_n}p_{n,m_n}G_{n,m_n+1} \end{array}}{G_1p_1G_2p_2\dots G_mp_mG_{m+1}}$$

Here $G_{i,j}$ $(i = 1,2,\dots,n;\ j = 1,2,\dots,m_i + 1)$ and $G_k$ $(k = 1,2,\dots,m + 1)$ are designations of certain concrete $H$-words, while $p_{i,j}$ $(i = 1,2,\dots,n;\ j = 1,2,\dots,m_i)$ and $p_k$ $(k = 1,2,\dots,m)$ are designations of certain schematic variables. Schematic variables are letters of some fixed alphabet $\{q, r, s, q', r', s', \dots\}$. It is assumed that every variable occurring in the conclusion of a certain production scheme occurs in at least one of the premises of this scheme.

The concept of "derivation in a given canonical calculus" (see ([1])) is defined in such a way that the admissible values of any schematic variable are arbitrary $H$-words, where $H$ is the alphabet of the given calculus. If, however, we take only $A$-words as admissible values of the variables, where $A \subset H$, then the same axioms and rules of inference will define a quasicanonical calculus in the alphabet $H$ with principal alphabet $A$. A canonical (quasicanonical) calculus $\mathfrak{R}$ is called a **calculus over the alphabet** $A$ if the alphabet of the calculus $\mathfrak{R}$ contains $A$. We shall say that the calculi $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are **equivalent relative**

**to the alphabet** $A$ if the $A$-word $S$ is derivable in $\mathfrak{R}_1$ if and only if it is derivable in $\mathfrak{R}_2$. Let $\mathfrak{P}$ be a canonical calculus over the alphabet $A$. We shall say that it is a **completely canonical calculus with principal alphabet** $A$ if $\mathfrak{P}$ is equivalent relative to $A$ to the quasicanonical calculus obtained from $\mathfrak{P}$ by fixing $A$ as the principal alphabet.

2. It is easy to construct a canonical calculus $\mathfrak{P}$ over the alphabet $A$ such that an $A$-word $S$ is derivable in this calculus if and only if the length of $S$ is expressed by a number of the form $2^{2^n}$ $(n = 0, 1, 2, ...)$. It can be shown that a canonical calculus in the alphabet $A$ producing all words of this type and only them is impossible. Consequently, **not for every canonical calculus over an alphabet** $A$ **does there exist a canonical calculus in the alphabet** $A$**, equivalent to the original one relative to** $A$.

* Words in the alphabet $H$ (including the empty word) will be called $H$-words.

At the same time the following theorems hold:

**Theorem 1.** Whatever the alphabet $A$ and the canonical calculus $\mathfrak{P}$ over $A$ may be, one can construct a canonical calculus $\mathfrak{D}$ in the alphabet $A \cup \{\xi\}$ $(\xi \notin A)$, equivalent to the calculus $\mathfrak{P}$ relative to $A$ and having the following properties: 1) $\mathfrak{D}$ is a completely canonical calculus with principal alphabet $A$; 2) every word $S$ derivable in $\mathfrak{D}$ contains at most one occurrence of the letter $\xi$, and in the case when $A$ has at least two letters, $S$ either is an $A$-word, or begins with the letter $\xi$ and contains no other occurrences of it; 3) $\mathfrak{D}$ has a single axiom and every production scheme of the calculus has only one premise.

**Theorem 2.** If $A$ is a one-letter alphabet, then, whatever the canonical calculus $\mathfrak{P}$ over $A$ may be, one can construct a canonical calculus $\mathfrak{D}$ in the alphabet $A \cup \{\xi\}$ $(\xi \notin A)$, equivalent to $\mathfrak{P}$ relative to $A$ and having the following properties: 1) $\mathfrak{D}$ is a completely canonical calculus with principal alphabet $A$; 2) every word derivable in $\mathfrak{D}$ either is an $A$-word or begins with the letter $\xi$ and contains no other occurrences of it; 3) $\mathfrak{D}$ has a single axiom and every production scheme of $\mathfrak{D}$ has no more than two premises.

In the case when the alphabet $A$ contains at least two letters, Theorem 1 is proved by a comparatively simple construction (using Post' s first reduction, see (1)). If $A = \{|\}$, Theorems 1 and 2, by means of Post' s theorem (see (1)), are reduced to the case when $\mathfrak{P}$ is a normal calculus in the alphabet $\{|, \alpha\}$; in the proof the following lemma is used:

**Lemma.** Let $A$ and $B$ be alphabets having no letters in common, $B = \{\xi, \xi_1, \xi_2, ..., \xi_L\}$. Let $\mathfrak{R}$ be a canonical calculus in the alphabet $A \cup B$ such that every axiom and every premise of any production scheme of this calculus contains exactly one occurrence of a letter from the alphabet $B$, while the conclusion of every scheme contains no more than one occurrence of a letter from this alphabet. Then one can construct a completely canonical calculus $\mathfrak{S}$ in the alphabet $A \cup \{\xi\}$ with principal alphabet $A$, equivalent to $\mathfrak{R}$ relative to $A$, and such that every word derivable in $\mathfrak{S}$ contains at most one occurrence of the letter $\xi$. Moreover, if $\mathfrak{R}$ is such that every word derivable in $\mathfrak{R}$ is an $A$-word or begins with a letter of the alphabet $B$, then $\mathfrak{S}$ has the same property.

Theorems 1 and 2 show that any algorithmically enumerable set of $A$-words $\mathfrak{M}$ can be specified by a canonical calculus $\mathfrak{D}$ with a single auxiliary letter $\xi$ such that, in an arbitrary derivation, the letter $\xi$ merely "distinguishes" the auxiliary derivable words from the basic words (i.e., the words belonging to $\mathfrak{M}$) and nowhere occurs as a delimiter.

**3.** A canonical calculus will be called **local** if it contains one axiom and every production scheme of this calculus has one of the following three forms:

$$\text{a) } \frac{qGr}{qG'r}, \qquad \text{b) } \frac{qG}{qG'}, \qquad \text{c) } \frac{Gr}{G'r}.$$

A local calculus will be called **narrowly local** (**superlocal**) if in any production scheme each of the words $G$ and $G'$ contains no more than two letters (respectively, no more than one letter). It is easy to see that for any canonical calculus in an alphabet $H$ one can construct a narrowly local calculus over $H$, equivalent to the original one relative to $H$. On the other hand, it can be proved that for every superlocal calculus there exists a decision algorithm. Therefore, not for every canonical calculus in an alphabet $H$ can one construct an equivalent, relative to $H$, superlocal calculus.

A production scheme $\Sigma$ of a local calculus $\Omega$ will be called **reversible** in $\Omega$ if $\Omega$ contains the scheme inverse to $\Sigma$, i.e. the scheme obtained–

from $\Sigma$ by a permutation of the premise and conclusion. A local calculus will be called **reversible** if all its schemes are reversible. Producing schemes of types b) and c) will be called **edge** schemes. A reversible local calculus that contains no edge schemes will be called an **associative calculus with a fixed axiom**. If $\mathfrak{A}$ is an associative calculus with a fixed axiom, then the list of schemes of this calculus will be called the **associative calculus corresponding to the given calculus** $\mathfrak{A}$, and will be denoted by $\widetilde{\mathfrak{A}}$. It is clear that $\widetilde{\mathfrak{A}}$ is an associative calculus in the generally accepted sense (see ([2])). The class of words derivable in $\mathfrak{A}$ coincides with the class of words equivalent, in the sense of ([2]), to the axiom of the calculus $\mathfrak{A}$ in the calculus $\widetilde{\mathfrak{A}}$.

**Theorem 3.** *Let $\Omega$ be a local calculus over the alphabet $A$. One can construct a local calculus $\Omega'$, containing no edge schemes and equivalent to the calculus $\Omega$ relative to $A$. The calculus $\Omega'$ can be constructed so that all schemes of $\Omega'$, except possibly one, are reversible.*

**Theorem 4.** *Let $\Omega$ be a local calculus over the alphabet $A$. One can construct a reversible calculus $\Omega'$, equivalent to $\Omega$ relative to $A$ and containing only four edge schemes.*

We give the construction of the calculus $\Omega'$. The set of $A$-words derivable in $\Omega$ is enumerable. Let $\mathfrak{U}$ be a normal algorithm in the alphabet $B$ ($A \cup \{\,|\,\} \subset B$), which transforms any word of the form $|^k$, to which it is applicable, into an $A$-word derivable in $\Omega$, and such that for every $A$-word $P$ derivable in $\Omega$ one can find an $l$ for which $\mathfrak{U}(|^l) = P$**. In a suitable extension of the alphabet $B$ one can construct an associative calculus $\mathfrak{B}$ such that, for any $B$-words $P$ and $Q$, the equality $\mathfrak{U}(P) = Q$ holds if and only if $\mathfrak{B} : \beta\alpha P\beta \parallel \beta\gamma Q\beta$ (see ([2]), p. 208). As the axiom of the calculus $\Omega'$ we take the word $\beta\delta_1|\beta$. As the producing schemes of the calculus $\Omega'$ we choose the schemes

$$\frac{q\beta\delta_1 \mid r}{q\beta\delta_1 \parallel r}, \qquad \frac{q\delta_1 \mid r}{q \mid \delta_1 r}, \qquad \frac{q\delta_1\beta r}{q\delta_2\beta r},$$

$$\frac{q \mid \delta_2 r}{q\delta_2 \mid r}, \qquad \frac{q\beta\delta_2 \mid r}{q\beta\alpha \mid r},$$

$$\frac{\beta\gamma q}{\eta q}, \qquad \frac{q\eta x r}{q x \eta r}\ (x \in A), \qquad \frac{q\eta\beta}{q},$$

the schemes inverse to them, and all schemes of the calculus $\mathfrak{B}$.

**Theorem 5.** *There exists a local calculus $\mathfrak{K}$ over the alphabet $A$ such that any reversible calculus equivalent to it relative to $A$ necessarily contains edge schemes both of type b) and of type c).*

As $\mathfrak{K}$ one may take, for example, a calculus over the alphabet $\{|\}$ such that a $\{|\}$-word $S$ is derivable in $\mathfrak{K}$ if and only if the length of $S$ is expressed by a number of the form $2^n$ ($n = 0, 1, 2, ...$). The proof rests on the obvious assertion: if a reversible calculus $\mathfrak{D}$ does not contain schemes b) (does not contain schemes c)), then, whatever the words $P, Q, R$ in the alphabet of the calculus may be, if $\mathfrak{D} : P \parallel Q$, then $\mathfrak{D} : PR \parallel QR$ (respectively $\mathfrak{D} : RP \parallel RQ$).

An immediate consequence of Theorem 5 is the following assertion:

*One can construct a canonical calculus over the alphabet $A$ for which there exists no associative calculus with a fixed axiom equivalent to it relative to $A$.*

* Two mutually inverse schemes for each edge of a word.

** $|^n$ denotes the word $\underbrace{||...|}_{n \text{ times}}$.

At the same time, it follows from the proof of Theorem 4 that *for any canonical calculus $\mathfrak{P}$ over the alphabet $A$, one can construct, over the alphabet $A \cup \{\bar{\beta}\}$ ($\beta \notin A$), an associative calculus $\mathfrak{D}$ with a fixed axiom such that the word $\beta S\beta$, where $S$ is an $A$-word, is derivable in $\mathfrak{D}$ if and only if $S$ is derivable in $\mathfrak{P}$.*

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*Note: Figure translations are in progress. See original paper for figures.*