A SYSTEM OF FIELD EQUATIONS FOR A NONIDEAL FERMI-DIRAC SYSTEM EXPLICITLY TAKING INTO ACCOUNT TWO-PARTICLE BOUND STATES

1962

SovietRxiv

Abstract

Full Text

PHYSICS

V. V. TOLMACHEV

A SYSTEM OF FIELD EQUATIONS FOR A NONIDEAL FERMI-DIRAC SYSTEM EXPLICITLY TAKING INTO ACCOUNT TWO-PARTICLE BOUND STATES

(Presented by Academician N. N. Bogolyubov on 29 V 1962)

In the preceding note (^{1}) we proposed a completely exact field relation connecting the one-particle and two-particle Green' s functions (equation (4) of (^{1})), which, when used to obtain field equations, leads to the appearance in them of an operation new for quantum statistics—taking the derivative of the Green' s function with respect to energy.

In (^{1}) two systems of field equations were formulated (equations (6), (7) and (10), (11)), which are essentially a generalization of the well-known Bethe-Salpeter field equation for two-particle bound states of two types (particle—particle or hole—hole, or else particle—hole). In these systems of equations the corresponding interaction operators I and J must be regarded as known (in fact, they must be replaced by the first order of perturbation theory for them). Such a procedure, however, excludes the possibility of revealing the influence of bound states of the two indicated types on one another. Therefore we are faced with the need to find some unified system of field equations for the joint study of two-particle bound states of both types.

Below we formulate such a system of equations. In it the interaction operators I and J are now unknown functions. The role of the operator characterizing the concrete kind of interaction is played by a new interaction operator T. It is precisely this new operator that we must now regard as known (replacing it by a new order of perturbation theory for it)*.

In the indicated system, two-particle states are fully taken into account explicitly, while three-particle and higher bound states are not taken into account, nor is their influence on the two-particle bound states of both types and on the structure of the one-particle Green' s function.

As in (^{1}), we have in mind a nonideal Fermi-Dirac system with a fourfold interaction, the diagram technique for which is sufficiently well known.

Let us introduce for consideration a new two-particle operator S

$$
\begin{aligned}
G(p_1E_1, p_2E_2;\, p'_1E'_1, p'_2E'_2) = {} & 2\pi\hbar\delta(E_1+E_2-E'_1-E'_2)\times \\
& \times \Delta(p_1+p_2-p'_1-p'_2)\, G(p_1E_1)\, G(p_2E_2) \\
& \times \{2\pi\hbar\delta(E_1-E'_1)\, \Delta(p_1-p'_1) \\
& \quad - 2\pi\hbar\delta(E_1-E'_2)\, \Delta(p_1-p'_2)\}+ \\
& + G(p_1E_1)G(p_2E_2)\, S(p_1E_1, p_2E_2;\, p'_1E'_1, p'_2E'_2)\, G(p'_1E'_1)G(p'_2E'_2).
\end{aligned}
\tag{1}
$$

Let us make use of the spatial and temporal homogeneity of the problem. Then

$$
\begin{aligned}
G(p_1E_1, p_2E_2;\, p'_1E'_1, p'_2E'_2) = {} & 2\pi\hbar\delta(E_1+E_2-E'_1-E'_2)\, \Delta(p_1+p_2- \\
& - p'_1-p'_2)\, \frac{1}{V}\, G(p_1-p'_1, p_1-p'_2, p_1+p_2;\, E_1-E'_1, E_1-E'_2, E_1+E_2),
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
S\,(p_1E_1, p_2E_2;\, p'_1E'_1, p'_2E'_2) = 2\pi\hbar\delta\,(E_1+E_2-E'_1-E'_2)\, \Delta\,(p_1+p_2 \\
- p'_1-p'_2)\, \frac{1}{V}\, S\,(p_1-p'_1, p_1-p'_2, p_1+p_2;\, E_1-E'_1, E_1-E'_2, E_1+E_2).
\end{aligned}
\tag{3}
$$

Using equation (4) from (^{1}) and (1), one can obtain the **first equation** of the desired system in the momentum-energy representation

$$
\begin{aligned}
\frac{\hbar}{i}\, \frac{\partial G(p_1E_1)}{\partial E_1} = G^2(p_1E_1) - \frac{1}{2\pi\hbar}\, \frac{1}{V} \sum_{p_2} \int_{-\infty}^{+\infty} dE_2\, G^2(p_1E_1)\, G^2(p_2E_2)\times \\
\times S\,(0, p_1-p_2, p_1+p_2;\, 0, E_1-E_2, E_1+E_2).
\end{aligned}
\tag{4}
$$

The S entering (4) should be understood, in the sense of the derivation of the equation, as the limiting value of S with nonzero first arguments (initially one obtains an equation in which, instead of the derivative, there stands a quotient of finite differences; only by letting these differences tend to zero do we arrive at (4)).

Eliminating from (5) and (8) from (^{1}) the two-particle Green’s function, expressing it by means of (1) through S, we obtain the **second and third equations** of the desired system in the momentum-energy representation:

---

* N. N. Bogolyubov drew the author’ s attention to the fact that, in essence, such an operator had already been considered earlier in field theory by Ter-Martirosyan (^{2}).

$$
\begin{aligned}
&S\,(p_1-p_1',p_1-p_2',p_1+p_2;\;E_1-E_1',E_1-E_2',E_1+E_2)=\\
&=-\frac{i}{\hbar}\,I\,(p_1-p_1',p_1-p_2',p_1+p_2;\;E_1-E_1',E_1-E_2',E_1+E_2)+\\
&+\frac{i}{\hbar}\,I\,(p_1-p_2',p_1-p_1',p_1+p_2;\;E_1-E_2',E_1-E_1',E_1+E_2)-\\
&-\frac{i}{\hbar}\,\frac{1}{V}\,\frac{1}{2\pi\hbar}\sum_{p_1''}\int_{-\infty}^{+\infty}dE_1''\,I\,(p_1-p_1'',p_1''-p_2,p_1+p_2;\;E_1-E_1'',E_1''-E_2,E_1+E_2)\times\\
&\times G(p_1''E_1'')\,G(p_1+p_2-p_1'',E_1+E_2-E_1'')\times\\
&\times S\,(p_1'-p_1',p_1''-p_2',p_1'+p_2';\;E_1'-E_1',E_1''-E_2,E_1'+E_2');
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
&S\,(p_1-p_1',p_1-p_2',p_1+p_2;\;E_1-E_1',E_1-E_2',E_1+E_2)=\\
&=-\frac{i}{\hbar}\,J\,(p_1-p_1',p_1-p_2',p_1+p_2;\;E_1-E_1',E_1-E_2',E_1+E_2)+\\
&+\frac{i}{\hbar}\,\frac{1}{V}\,\frac{1}{2\pi\hbar}\sum_{p_1''}\int_{-\infty}^{+\infty}dE_1''\,J\,(p_1-p_1',p_1-p_1'',p_1'+p_1'';\;E_1-E_1',E_1-E_1'',E_1'+E_1'')\times\\
&\times G(p_1''E_1'')\,G(p_1'+p_1''-p_1,E_1'+E_1''-E_1)\times\\
&\times S\,(p_1-p_1',p_1''-p_2',p_1'+p_2'';\;E_1-E_1',E_1''-E_2',E_1'+E_2'').
\end{aligned}
\tag{6}
$$

In (5) and (6), similarly to (2) and (3), we have used the spatial and temporal homogeneity of the operators I and J.

$$
\begin{aligned}
&I\,(p_1E_1,p_2E_2;\;p_1'E_1',p_2'E_2')=2\pi\hbar\delta\,(E_1+E_2-E_1'-E_2')\,\Delta\,(p_1+p_2\\
&-p_1'-p_2')\,I\,(p_1-p_1',p_1-p_2',p_1+p_2;\;E_1-E_1',E_1-E_2',E_1+E_2);
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&J\,(p_1E_1,p_2E_2;\;p_1'E_1',p_2'E_2')=2\pi\hbar\delta\,(E_1+E_2-E_1'-E_2')\,\Delta\,(p_1+p_2\\
&-p_1'-p_2')\,J\,(p_1-p_1',p_1-p_2',p_1+p_2;\;E_1-E_1',E_1-E_2',E_1+E_2).
\end{aligned}
\tag{8}
$$

Finally, we present the **fourth** equation of the system. Introduce a new operator T according to the relation

Figure 1: Reducibility of diagrams by types I, II, III

Figure 1: Figure 1: Reducibility of diagrams by types I, II, III

$$
\begin{aligned}
&2S(p_1-p_1',p_1-p_2',p_1+p_2;E_1-E_1',E_1-E_2',E_1+E_2)=\\
&\quad=\frac{i}{\hbar}T(p_1-p_1',p_1-p_2',p_1+p_2;E_1-E_1',E_1-E_2',E_1+E_2)-\\
&\qquad-\frac{i}{\hbar}I(p_1-p_1',p_1-p_2',p_1+p_2;E_1-E_1',E_1-E_2',E_1+E_2)+\\
&\qquad+\frac{i}{\hbar}I(p_1-p_2',p_1-p_1',p_1+p_2;E_1-E_2',E_1-E_1',E_1+E_2)-\\
&\qquad-\frac{i}{\hbar}J(p_1-p_1',p_1-p_2',p_1+p_2;E_1-E_1',E_1-E_2',E_1+E_2)+\\
&\qquad+\frac{i}{\hbar}J(p_1-p_2',p_1-p_1',p_1+p_2;E_1-E_2',E_1-E_1',E_1+E_2).
\end{aligned}
\tag{9}
$$

Considering the operator T as a known function, the system of four equations (4), (5), (6), (9) may be understood as a system of equations for the four unknown functions G, I, J, S.

Equation (4) is sufficiently explained in (1). Equations (5), (6) are, in essence, relations known in field theory, on which the Bethe–Salpeter equation is based. We shall therefore dwell on explaining equation (9) (in this connection, however, see also (2)).

The diagrams for I, J, S, T are connected diagrams, without vacuum loops, with two incoming and two outgoing ends, from whose four ends the self-energy parts cannot be "pulled out." The diagrams for $S(12;1'2')$ are any such diagrams. The diagrams for $I(12;1'2')$ cannot be divided into two parts in such a way that one part contains $1, 2$, and the other $1', 2'$ (moreover, in these diagrams, starting from $1'(2')$, one can reach $1(2)$ only along electron lines). The diagrams for $J(12;1'2')$ cannot be divided into two parts in such a way that one part contains $1, 1'$, and the other $2, 2'$. It is also convenient to consider diagrams for $I'(12;1'2') = -I(12;2'1')$ and for $J'(12;1'2') = -J(12;2'1')$. The diagrams for I' have the same property as the diagrams for I (only for them, conversely, starting from $1'(2')$, one can reach $2(1)$ only along electron lines). The diagrams for J' cannot be divided into two parts such that one part contains $1, 2'$, and the other $1', 2$.

Fig. 1. Reducibility of diagrams by types I, II, III

Thus, the diagrams S may, generally speaking, be reducible* by type I, II, III according to the schemes (Fig. 1). The diagrams $I + I'$ are diagrams S not reducible by type I. The diagrams J are diagrams S not reducible by type II. The diagrams J' are diagrams S not reducible by type III.

---

* It should be noted that, in using below the notion of reducibility, we do not have in mind any topological procedure of reducing diagrams to certain "skeleton" diagrams (as is the case when excluding self-energy parts from diagrams by means of the mass operator Λ). In fact, no skeleton diagrams exist in the case under consideration. Reducibility is simply the representability of diagrams by type I, II, or III. Here we mean "skeleton" diagrams in which, instead of the fourth vertex, there would stand I or J.

In the case of a four-leg interaction we have been able to find a proof of the following assertion. There exists no diagram that is simultaneously reducible with respect to any two of the three types I, II, III. We do not give the proof here for lack of space.

Using the assertion formulated, we can carry out the following classification of diagrams into mutually nonintersecting classes. Diagrams S: either reducible by type I (these are diagrams $S - I - I'$); or reducible by type II (these are diagrams $S - J$); or reducible by type III (these are diagrams $S - J'$); or irreducible simultaneously with respect to all three types (these are diagrams T). Thus:

$$(S - I - I') + (S - J) + (S - J') + T = S \tag{10}$$

or

$$2S = -T + I + I' + J + J'. \tag{11}$$

In expanded form (11) is nothing other than (9).

There are 2 diagrams S in first order, 10 in second order, and 82 in third order (not counting the self-energy parts of first order). There are 2 diagrams T in first order, none in second and third order; however, already in fourth order there are diagrams T. The diagrams for I number 1 in first order, 4 in second order, and $32 + 14 = 46$ in third order (14 with self-energy insertions of first order); the diagrams for J number 2 in first order, 6 in second order, and $50 + 14 = 64$ in third order (14 with self-energy insertions of first order).

It seems to us that the system of exact field equations obtained in the proposed note may, in a certain sense, be regarded as a generalization of the known Schwinger system of equations $^{(3)}$. The latter is adapted to the study of one-particle excitations of the field without relation to the possible presence in the field of two-particle, three-particle, etc., excitations (the vertex operator Γ is replaced by first order of perturbation theory). Our system is adapted to the study of both one-particle and two-particle excitations of the field, but now already in connection with the possible presence in the field of three-particle, etc., excitations (the operator T must be replaced by first order of perturbation theory).

The author expresses gratitude to Academician N. N. Bogolyubov for a useful discussion of the work, and also to K. A. Ter-Martirosyan for a useful conversation.

Received
26 V 1962

Note added in proof. D. A. Kirzhnits kindly drew the author' s attention to the fact that the exact field relation of equation (4) from (1) is analogous to the Ward identity. Being written in finite differences, it is a broader relation. In quantum statistics the Ward identity was used in the theory of Fermi liquids by L. D. Landau (4). In addition, in (4) the operator S had already been introduced into consideration earlier.

CITED LITERATURE

[1] V. V. Tolmachev, DAN, **144**, No. 5 (1962).
[2] K. A. Ter-Martirosyan,
[3] J. Schwinger, Proc. Nat. Acad. Sci. USA, **37**, 452, 455 (1951).
[4] L. D. Landau, ZhETF **35**, 97 (1958); L. P. Pitaevskii, ZhETF, **37**, 1794 (1959).

Note: Figure translations are in progress. See original paper for figures.