SovietRxiv

Soviet-era science, translated into English

MATHEMATICS

S. M. NIKOLSKII

1962

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196201.73941

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

MATHEMATICS

S. M. NIKOLSKII

ON THE BOUNDARY PROPERTIES OF DIFFERENTIABLE FUNCTIONS OF SEVERAL VARIABLES

(Presented by Academician S. L. Sobolev on 16 IV 1962)

1. In this note we consider a function $\Phi(\bar{x})$, defined in some domain $G \subset R_n$ with boundary Γ, concerning some partial derivatives of which it is known that they have finite norm in the sense of $L_p(G)$. The question is posed as to which of its partial derivatives have meaningful stable limiting values on Γ—boundary functions—and in what sense they should be understood. In the case of functions f of such classes as $W_p^{(r)}$, $H_p^{(r)}$, $B_p^{(r)}$, $\cdots$, when the differential properties of f are the same in all directions, this question is at present well studied for sufficiently good domains. For the classes $W_p^{(r_1,\dots,r_n)}$, $H_p^{(r_1,\dots,r_n)}$ it is also rather well studied if the boundary Γ is a coordinate plane of one or another dimension or a part of it. (For literature on this question, see, for example, our survey ([2]).) But in the case of arbitrary domains, even with a very good boundary, the question has been very little studied and requires investigation.

In the present and subsequent notes this question will be considered for classes broader than $W_p^{(r)}$ and $W_p^{(r_1,\dots,r_n)}$. We shall also dwell on applications in the theory of boundary-value problems for equations of hypoelliptic type and even equations going beyond this type, but in a certain sense standing on the verge of this type.

2. Let us agree to denote by

$$\Delta = \{a_s < x_s < b_s;\ s = 1, \dots, n\} \tag{1}$$

a rectangular parallelepiped with edges parallel to the coordinate axes of R_n. We fix Δ and let $\Delta_i = \mathrm{pr}_i\,\Delta$ be the projection of Δ onto the plane $x_i = 0$. On Δ_i a uniformly continuous bounded function is given

$$x_i = \psi(\bar{y}), \qquad \bar{y} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \Delta_i, \tag{2}$$

for which $b_i < \psi(\bar{y})$ (or $\psi(\bar{y}) < a_i$) on Δ_i. The set Λ of points $\bar{x} = (x_i, \bar{y})$, for which the inequality

SovietRxiv

$$a_i < x_i < \psi(\bar{y}) \qquad (\bar{y} \in \Delta_i)$$

(or $\psi(\bar{y}) < x_i < b_i$) is satisfied, will be called an x_i-domain. Thus, Λ is a domain (an open connected set), it contains Δ and $\mathrm{pr}_i\,\Delta = \mathrm{pr}_i\,\Lambda$. The part γ of the boundary of Λ described by equation (2) is called the x_i-boundary of Λ.

Theorem 1. Let Λ be an x_i-domain, $\Delta \subset \Lambda$, $\mathrm{pr}_i\,\Delta = \mathrm{pr}_i\,\Lambda = \Delta_i$, and

$$\|f\|_{L_p(\Delta)} < \infty, \qquad \left\|\frac{\partial^r f}{\partial x_i^r}\right\|_{L_p(\Lambda)} < \infty \qquad (1 \le p \le \infty).$$

Then

$$\left\|\frac{\partial^l f}{\partial x_i^l}\right\|_{L_p(\Lambda)} \le c\left(\|f\|_{L_p(\Delta)} + \left\|\frac{\partial^r f}{\partial x_i^r}\right\|_{L_p(\Lambda)}\right) \qquad (l = 0, 1, \dots, r-1). \tag{3}$$

and on the x_i-boundary γ of the domain Λ there exist x_i-limit (boundary) functions

$$\mu_l(\bar{y}) = \lim_{x_i \to \psi(\bar{y})} \frac{\partial^l f}{\partial x_i^l}(x_i, \bar{y}), \qquad (l = 0, 1, \dots, r-1); \tag{4}$$

in the sense of convergence almost everywhere and in the mean in the sense of $L_p(\Delta_i)$:

$$\|\mu_l(\bar{y}) - f_{x_i}^{(l)}(\psi(\bar{y}) - \varepsilon, \bar{y})\|_{L_p(\Delta_i)} \to 0 \qquad (\varepsilon \to 0).$$

Moreover,

$$\|\mu_l\|_{L_p(\Delta_i)} \le c\left(\|f\|_{L_p(\Delta)} + \left\|\frac{\partial^r f}{\partial x_i^r}\right\|_{L_p(\Delta)}\right), \tag{5}$$

where the constant c, here and below, does not depend on a number of quantities.

This theorem is obtained on the basis of the study of Taylor' s formula for f in powers of $x_i - x_i^0$. Equality (4) for $r = 1$ was obtained by L. D. Kudryavtsev ([1]) for the somewhat more general sets of x_i-intervals introduced by him, rather than the sets Λ.

3. Let $e_n = \{1, \dots, n\}$; let e be any subset of it, in particular the empty one, and if $r = (r_1, \dots, r_n)$ is a nonnegative integer vector ($r_i \ge 0$ integers), then $r^e = (r_1^e, \dots, r_n^e)$ is the vector where $r_s^e = r_s$ if $s \in e$, and $r_s^e = 0$ if $s \in e_n - e$.

SovietRxiv

We introduce the norm

$$\|f\|_{S_p^{(r)}(G)} = \sum_{e\subset e_n} \|f^{(r^e)}\|_{L_p(g)}, \qquad (1 \leq p \leq \infty), \tag{6}$$

where

$$f^{(k)} = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n} f}{\partial x_n^{k_n}} \qquad (k = (k_1, \dots, k_n)) \tag{7}$$

with the indicated order of differentiation, and we introduce the (Banach) space $S_p^{(r)}(G)$ of functions defined on G with finite norm (6).

Let Ω be an x_i-domain simultaneously for all $i = 1, \dots, n$, or let Ω be an x_i-domain only for $i = 1, \dots, m < n$, while in the remaining directions $x_{m+1}, \dots, x_n$ it has a cylindrical character. The simplest set Ω is Δ. The inequality

$$\|f^{(\vec{\rho})}\|_{L_p(\Omega)} \leq c\|f\|_{S_p^{(r)}(\Omega)} \qquad (0 \leq \vec{\rho} \leq r) \tag{8}$$

is valid.

Here $f^{(\vec{\rho})}$ (and $f^{(r)}$) does not depend on the order of differentiation. From (8) and Theorem 1 it follows:

Lemma 1. If $k = (k_1, \dots, k_n) < (k_1, \dots, k_{i-1}, k_i + 1, k_{i+1}, \dots, k_n) \leq r$, $f \in S_p^{(r)}(\Omega)$, then the derivative $f^{(k)}$ has on the x_i-boundary γ of the domain Ω an x_i-boundary function μ with norm

$$\|\mu\|_{L_p(\Omega)_i} \leq c\|f\|_{S_p^{(r)}(\Omega)}, \tag{9}$$

where $\Omega_i = \operatorname{pr}_i \Omega$.

4. If a function f simultaneously belongs to $S_p^{(k^s)}(\Delta)$, where k^s are given $(s = 1, \dots, N)$ nonnegative integer vectors, $k = \sum_1^N \lambda_s k^s$,

$$\sum_1^N \lambda_s \leqslant 1, \quad \lambda_s \geqslant 0,$$

then*

$$\|f\|_{S_p^{(\mathbf{k})}(\Delta)} \leqslant c \sum_1^N \|f\|_{S^{(\mathbf{k}^s)}(\Delta)} \tag{10}$$

(see (1)).

SovietRxiv

5. By definition, a point $\bar{x}_0$ of the boundary Γ of a domain G is called regular if one can specify a rectangular parallelepiped Δ_0, containing $\bar{x}_0$ strictly inside it, such that $G\Delta_0$ is an Ω (see item 2 above, (8)). The remaining points of Γ will be called exceptional. By definition, G is a regular domain if the projection of the set of exceptional points of its boundary onto any plane $x_i = 0$ has $(n-1)$-dimensional measure zero. In the case of a smooth boundary Γ, a point $\bar{x}_0$ of it at which the tangent plane is not parallel to any of the coordinate axes is regular. If a piece σ of the boundary Γ in a neighborhood of $\bar{x}_0$ can be written in the form $\psi(x_1, \dots, x_m) = 0$ $(m < n)$, where ψ is a continuously differentiable function and the tangent plane at $\bar{x}_0$ is not parallel to the axes $x_1, \dots, x_m$, then $\bar{x}_0$ is also a regular point. Domains with piecewise smooth boundary are regular.

6. Let $\mathcal{E}$ be a bounded set of nonnegative integer vectors, convex in the sense that the totality of integer vectors belonging to the least convex body $\overline{\mathcal{E}}$ containing $\mathcal{E}$ coincides with $\mathcal{E}$, and let $\mathcal{E}$ certainly contain the vectors

$$\vec{\omega}_1 = (r_1, 0, \dots, 0), \dots, \vec{\omega}_n = (0, \dots, 0, r_n)$$

$(r_j > 0)$. In addition, if $\mathbf{k} \in \mathcal{E}$, then $\mathbf{k}^e \in E$ for all $e \subset e_n$. Further, let $0, \mathbf{k}^1, \dots, \mathbf{k}^N$ be the supporting vectors of $\mathcal{E}$, forming the smallest system of vectors such that for any $\mathbf{k} \in \mathcal{E}$ there is a representation

$$\mathbf{k} = \sum_1^N \lambda_s \mathbf{k}^s, \quad \sum_1^N \lambda_s \leqslant 1, \quad \lambda_s \geqslant 0$$

(not in general unique).

Theorem 2. *Let a function Φ be given in a domain $G \supset R_n$, for which*

$$D_G(\Phi) = \int_G \sum_1^N |\Phi^{(\mathbf{k}^s)}|^q \, dG < \infty. \tag{11}$$

Then for any $\Delta \subset G$ (see (1)) the norm $\|\Phi\|_{L_p(\Delta)} < \infty$, and for any $\mathbf{k} \in \mathcal{E}$

$$\|\Phi^{(\mathbf{k})}\|_{L_p(\Delta)} \leqslant c_\Delta (D_G(\Phi) + \|\Phi\|_{L_p(\Delta)}). \tag{12}$$

For the domain $\Omega = G\Delta_0$, defined in item 4 and adjacent to a regular point $\bar{x}_0$ of the boundary Γ, and for every $\mathbf{k}$ such that $\mathbf{k} \leqslant \mathbf{k}^s$ for some $s = 1, \dots, N$, the norm $\|\Phi\|_{L_p(\Omega)} < \infty$ and

$$\|\Phi^{(\mathbf{k})}\|_{L_p(\Omega)} \leqslant C_\Omega (D_G(\Phi) + \|\Phi\|_{L_p}). \tag{13}$$

Moreover, if $\gamma = \Gamma\Delta_0$ is the x_i-boundary of Ω and

$$\mathbf{k} = (k_1, \dots, k_n) < (k_1, \dots, k_{i-1}, k_i + 1, k_{i+1}, \dots, k_n) \leqslant \mathbf{k}^s$$

SovietRxiv

for some $s = 1, \dots, N$, *then* $f^{(\mathbf{k})}$ *on* γ *has the* x_i*-boundary function* $\mu_{k,\gamma,i} = f_\Gamma^{(\mathbf{k})|(i)}$, *for which*

$$\|\mu_{k,\gamma,i}\|_{L_p(\Omega_i)} \leqslant c\big(D_G(\Phi) + \|\Phi\|_{L_p(\Omega)}\big). \tag{14}$$

Thus, every function Φ with $D_G(\Phi) < \infty$ has a definite set of boundary functions μ, which in any case possess local

---

* In obtaining this inequality I used a theorem (unpublished) of Yu. L. Bessonov, with his consent.

stability (on Ω). One can indicate minimal such sets that completely determine the whole set. The vectors

$$0 < l^1 < l^2 < \dots < l^\nu < k$$

by definition form a chain suitable for k if l^j differs from l^{j+1} in only one component, and this component in l^{j+1} is greater by 1 than in l^j. The vector k itself is not included in the chain. Put into correspondence with each k^s one of its chains. For different s they may intersect, but not completely. We shall call the skeleton $\mathcal{E}$ the set $S_\mathcal{E}$ consisting of the vectors that belong to all the chains k^s $(s = 1, \dots, N)$. If $k \in S_\mathcal{E}$, then k may be immediately followed by several vectors k' belonging to $S_\mathcal{E}$ (differing from k in only one component: $k'_j = k_j + 1$).

The totality of boundary functions of the derivatives $f^{(k)}$, where $k \in S_\mathcal{E}$, forms a minimal set of boundary functions of the function Φ. Here it is counted that, if several vectors $k' \in S_\mathcal{E}$ immediately follow k, then the set includes all x_j-boundary functions of $f^{(k)}$ for those j for which $k'_j = k_j + 1$.

Theorem 3. *If two functions* Φ_1, Φ_2, *for which (11) holds, possess one and the same (corresponding to a definite skeleton* $S_\mathcal{E}$*) set of boundary functions, then they both have coinciding boundary functions for any* k *for which* $k < k^s$ *for some* $s = 1, \dots, N$. *For the remaining* $k < k' \in \mathcal{E}$ *the boundary functions for* $\Phi_1^{(k)}$, $\Phi_2^{(k)}$, *if they exist, coincide simultaneously and correspondingly.*

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
7 IV 1962

CITED LITERATURE

1. L. D. Kudryavtsev, *Scientific Reports of Higher School, Phys.-Math. Sciences*, No. 3, 25 (1959).

2. S. M. Nikol' skii, *UMN*, 16, 5(101), 63 (1961).

3. S. M. Nikol' skii, *Izv. AN SSSR, Ser. Math.*, 23, 213 (1959).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.