MATHEMATICS

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1962

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Abstract

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RADICALS AND THE DECOMPOSITION OF A RING

In the present note it is proved that Faith' s theorem (¹) on the decomposition of an associative MP-ring into a direct sum of a regular radical and a ring bounded by the Jacobson radical is a special case of a more general proposition.

Associative rings are considered. By the radical of a ring we mean an axiomatic radical in the sense of Kurosh (²). The radical of a ring is called **hereditary** if every ideal of a radical ring is a radical ring. As is known (³), heredity of a radical is equivalent to the following property:

If $R(K)$ is the radical of an arbitrary ring K, then for every ideal B of K the equality

$$R(B) = B \cap R(K). \tag{1}$$

holds.

A ring K is called **strongly R-semisimple**, where R is some radical, if every homomorphic image of it is an R-semisimple ring. Let R be the given radical. The radical R is called **complemented to** R if R_c is the largest among all radicals having, in every ring, zero intersection with R. A hereditary radical R is called **supernilpotent** (⁴) if nilpotent rings are R-radical, i.e., if in every ring K the radical $R(K)$ contains all nilpotent ideals of the ring K. The known radicals of Baer, Levitzki, Jacobson, Brown—McCoy, and others are supernilpotent radicals. The smallest supernilpotent radical is the Baer radical L, since it is known that the class of all L-semisimple rings coincides with the class of all rings without nilpotent ideals.

A ring K is called **hereditarily idempotent** (**f-regular**) (⁵) if every ideal of it is idempotent.

A hereditary radical R is called **subidempotent** (⁴) if the R-radical rings are idempotent. The largest subidempotent radical is the hereditarily idempotent (f-regular) radical of Blair F (⁵).

Proposition 1 (⁴), Theorems 3, 4). *If R is a supernilpotent radical, then there exists a radical R_c complementary to it. The radical R_c will be a subidempotent* *radical, and the R_c-radical rings are precisely the strongly R-semisimple rings.*

If B is an ideal of the ring K, then B^* denotes the annihilator of the ideal B, i.e., $B^* = \{a \in K \mid aB = Ba = 0\}$. As usual, we shall say that a ring K is bounded by the radical R if $R^* \subseteq R$. We note that if the ring K is bounded by the radical R, then it will be bounded by any radical R_1 containing R. A ring with the minimality condition for principal right ideals will briefly be called an MP-ring. Kaplansky (^{6}) observed that every homomorphic image of an MP-ring is an MP-ring.

The aim of the present note is to prove the following theorem:

Theorem. *Let $R = R(K)$ be the given supernilpotent radical of the ring K. If $\overline{K} = K/R$ is an MP-ring, then*

$$K = R_c \dotplus R_c^*, \tag{2}$$

where either $R_c = 0$, or R_c is a discrete direct sum of R-semisimple

simple MP-rings. Moreover, $R(R_c^*) = R(K)$ and the ring R_c^* is bounded by the radical R. Finally, either $R_c^* = R$, or R_c^*/R is a discrete direct sum of R-semisimple MP-rings.

For the proof of the theorem we shall need a number of concepts and auxiliary propositions. Recall that a ring K is called weakly regular (^{7}) if for every element a of K there exists an element a_1 in the principal ideal (a) such that $a = aa_1$. It is not difficult to show that the ring K is weakly regular if and only if each of its right ideals is idempotent. Every ideal of a weakly regular ring is a weakly regular ring. A weakly regular ring is, obviously, a hereditarily idempotent ring. The converse assertion is, in general, false.

Lemma 1 (see (^{8}), Lemma 12). *If Q is an ideal of a ring K such that the factor ring $\overline{K} = K/Q$ is weakly regular, then*

$$Q \cap (Q^*)^2 = 0.$$

Proposition 2. *Let R be a given supernilpotent radical of a ring K, and suppose that K/R is weakly regular and at the same time a strongly R-semisimple ring. Then the equality $R_c = 0$ is equivalent to the inclusion $R^* \subseteq R$.*

Proof. By Proposition 1, the supplementary radical R_c exists. Let us first note that $R^* \subseteq R$ always implies $R_c = 0$. Indeed, since $R_c \cap R = 0$, we have $R_{cR} = RR_c = 0$, whence $R_c \subseteq R^* \subseteq R$. Consequently, $R_c = R_c \cap R = 0$. Now let K/R be a weakly regular and at the same time strongly R-semisimple ring. By Lemma 1,

$$R \cap (R^*)^2 = 0,$$

whence

$$\frac{(R^*)^2 + R}{R} \cong (R^*)^2.$$

On the left stands an ideal of a strongly R-semisimple, i.e., in view of Proposition 1, R_c-radical, ring K/R. By the same Proposition 1,

$$\frac{(R^*)^2+R}{R}$$

and, consequently, $(R^*)^2$, will be R_c-radical rings. Therefore $(R^*)^2 \subseteq R_c$. On the other hand, $R_c \subseteq R^*$, whence $R_c^2 \subseteq (R^*)^2$. But, by Proposition 1, $R_c^2 = R_c$, and therefore $R_c \subseteq (R^*)^2$. Consequently, $R_c = (R^*)^2$. Hence, from Proposition $R_c = 0$ we obtain $(R^*)^2 = 0$. Since the radical R contains all nilpotent ideals of the ring, it follows that $R^* \subseteq R$.

Recall that a ring K is called regular in the sense of Neumann if for every element a of K there exists an $x \in K$ such that $a = axa$. Every ideal of a regular ring is a regular ring. A regular ring is a weakly regular ring. In the class of commutative rings the notions of regular, weakly regular, and hereditarily idempotent rings coincide.

Proposition 3. *An R-semisimple MP-ring K, where R is a supernilpotent radical, is a discrete direct sum of R-semisimple simple MP-rings.*

Indeed, as is known (see ([9]), Theorem 17), every MP-ring without nilpotent ideals is a discrete direct sum of simple idempotent MP-rings B_α, i.e.

$$K = \sum_\alpha B_\alpha.$$

By virtue of equality (1),

$$R(B_\alpha) = B_\alpha \cap R(K) = 0.$$

Proposition 4 (see ([10])). *A ring K is a discrete direct sum of simple rings if and only if every two-sided ideal of the ring K is a direct summand in K. Every ideal of the indicated ring is a ring of the same type.*

Lemma 2. *If $K = A+B$, where A and B are ideals of the ring K, then $B = A^*$ if and only if $A^*_A = 0$ (A^*_A is the annihilator of the ideal A in the ring A).*

Indeed, since $B \subseteq A^*$, we have

$$A^* = A^* \cap (A+B) = A^* \cap A + B = A^*_A + B.$$

Proof of the theorem. By Proposition 1, the supplementary radical R_c exists. Let

$$\overline{R_c} = \frac{R_c+R}{R}$$

be the image of the ideal

R_c under the natural homomorphism $K \to \overline{K} = \dfrac{K}{R}$. Note that, since $R_c \cap R = 0$, we have $\overline{R}_c \simeq R_c$. Since $\overline{K}$ is an R-semisimple MP-ring, by Proposition 3,

$\overline{K}$ will be a discrete direct sum of R-semisimple simple MP-rings. In view of Proposition 4, $\overline{K}$ is a strongly R-semisimple ring. Since, by the same Proposition 4, the ideal $\overline{R}_c$ splits off as a direct summand in the ring $\overline{K}$, there exists in K an ideal $S \supseteq R$ such that $K = R_c + S$ and $R_c \cap S \subseteq R$. Since $R_c \cap R = 0$, from the last inclusion we obtain $R_c \cap S = 0$. Consequently, $K = R_c \dotplus S$, and since the condition of Lemma 2 is fulfilled for the hereditarily idempotent ring R_c, we have $S = R_c^*$. Thus $K = R_c \dotplus R_c^*$. Since $R_c \simeq \overline{R}_c$, R_c will be a discrete direct sum of R-semisimple simple MP-rings. From the relations $R \subseteq S = R_c^*$ it follows that $R(R_c^*) = R_c^* \cap R(K) = R$.

We now prove that the ring R_c^* is bounded by the radical R. If $R_c^* = R$, then there is nothing to prove. Now let $R_c^* \neq R$. Since

$$K = \overline{R}_c \dotplus \frac{R_c^*}{R}$$

is an MP-ring, $\dfrac{R_c^*}{R}$ will also be an MP-ring. By Proposition 3, the ring $\dfrac{R_c^*}{R}$ will be a discrete direct sum of simple rings with minimal right ideals, i.e. regular rings. Therefore $\dfrac{R_c^*}{R}$ is a regular ring. Since $R_c(R_c^*) = R_c^* \cap R_c(K) = 0$, by Proposition 2 the ring R_c^* will be bounded by the radical R. The theorem is proved.

Let us consider some special cases of the theorem proved.

I. Let $R = L$, where L is the Baer radical (13). Recall that the Baer radical is the smallest nil ideal S of the ring K with the property that K/S is a ring without nilpotent ideals. In the present case the complementary radical R_c coincides with the hereditarily idempotent Baer radical F (see (12)). Since every idempotent simple ring is L-semisimple, in this case the theorem proved is formulated as follows:

If $\overline{K} = K/L$ is an MP-ring, then $K = F \dotplus F^*$, where either $F = 0$, or F is a discrete direct sum of idempotent simple MP-rings. Moreover, $L(F^*) = L(K)$ and the ring F^* is bounded by the radical L. Finally, either $F^* = L$, or F^*/L is a discrete direct sum of idempotent simple MP-rings.

Note that F^* is an F-semisimple ring, i.e. a subdirect sum of subdirectly irreducible rings with nilpotent heart (12). From the mentioned proposition it follows:

Corollary 1. Every F-semisimple MP-ring is bounded by the Baer radical. In particular, every subdirectly irreducible MP-ring with nilpotent heart is bounded by the Baer radical.

Indeed, if $F = 0$, then $K = F^*$.

If M_r is the regular radical of the ring (11), then, generally speaking, $M_r \subset F$.

Corollary 2. If $\overline{K} = K/L$ is an MP-ring, then $F = M_r$.

Indeed, a discrete direct sum of idempotent simple MP-rings is a regular ring and, consequently, $F \subset M_r$.

II. Let $R = J$, where J is the Jacobson radical (7). If M_s is the weakly regular radical (7), then $M_s \subset J_c \subset F$. Indeed, it is easy to verify that $M_s \cap J = 0$, whence $M_s \subseteq J_c$. Since the complementary radical J_c is a hereditarily idempotent ring, $J_c \subseteq F$. Moreover, as Sasiada showed (14), there exists an idempotent simple ring, whose radical ···

in the sense of Jacobson. Consequently, $J_c \neq F$. For the case $R = J$ the theorem was proved by Faith (1) and can be formulated as follows:

If $\overline{K} = K/J$ *is an* MP*-ring, then* $K = J_c \dotplus J_c^*$*, where either* $J_c = 0$*, or* J_c *is a discrete direct sum of idempotent simple* MP*-rings. Moreover,* $J(J_c^*) = J(K)$*, and the ring* J_c^* *is bounded by the radical* J*. Finally, either* $J_c^* = J$*, or* J_c^*/J *is a discrete direct sum of idempotent* MP*-rings.*

It follows from the last sentence that if $\overline{K} = K/J$ is an MP-ring, then $J_c = M_r = M_s$, where M_r, as in the preceding case, is the regular radical.

III. Let now $R = N$, where N is the generalized nilradical of the ring (12). Recall that generalized nilrings are rings that are not mapped homomorphically onto rings without zero divisors. Commutative generalized nilrings are precisely nilrings. N-semisimple rings coincide with subdirect sums of rings without zero divisors. An N-semisimple ring K will be an MP-ring if and only if K is a discrete direct sum of fields. Recall that a ring K is called strictly regular if for every element a of K there is an $x \in K$ such that $a = a^2x$. If M_t is the strictly regular radical of the ring (15), then it is easy to show that $M_t \cap N = 0$. Hence $M_t \subseteq N_c$. In the decomposition (2)

$$K = N_c \dotplus N_c^*,$$

N_c will be a discrete direct sum of fields and, consequently, a strictly regular ring. Therefore, if K/N is an MP-ring, then $N_c = M_t$.

Let now in the ring K the minimum condition be satisfied for all right ideals, and let the supernilpotent radical R be such that it induces the classical nilpotent radical (for example, as in the cases I and II considered, but not III). Then equality (2) gives the well-known Wedderburn decomposition (16) of the ring K into the direct sum of a classical semisimple ring and a ring bounded by its nilpotent radical.

In conclusion we note that the following two-sided analogue of Faith' s theorem holds.

If in the ring $\overline{K} = K/T$*, where* T *is the Brown-McCoy-Sasiada radical* (17)*, the minimum condition for principal two-sided ideals is satisfied, then* $K = T_c \dotplus T_c^*$*, where either* $T_c = 0$*, or* T_c *is a discrete direct sum of simple rings with identity. Moreover,* $T(T_c^*) = T(K)$*, and the ring* T_c^* *is bounded by the radical* T*. Finally, either* $T_c^* = T$*, or* T_c^*/T *is a discrete direct sum of simple rings with identity.*

Let us note that in the decomposition $K = T_c \dotplus T_c^*$ the summand $T_c = B$, where B is the maximal biregular ideal of the ring K (18).

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Received
3 IV 1962

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