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Abstract

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ON THE METHOD OF POTENTIALS FOR A PARABOLIC EQUATION WITH DISCONTINUOUS COEFFICIENTS

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The paper considers the solution of the principal boundary-value problems, as well as mixed problems with an unbounded domain of prescription of the initial data (including the Cauchy problem), for a one-dimensional parabolic equation with coefficients having discontinuities of the first kind along a finite number of lines. With respect to the lines of discontinuity, as well as the lateral boundaries, it is assumed that they are given by equations of the form $x = X(t)$, where $X(t)$ satisfies, in t, a Hölder condition with exponent $> 1/2$. The coefficients of the equation outside the lines of discontinuity must, moreover, satisfy a Hölder condition in x (for the leading coefficient also in t) with nonzero exponents. In domains unbounded in x, the initial functions and the right-hand side may have exponential growth in x. The paper proves the existence of classical solutions of such problems, continuous together with the first derivative with respect to x up to the lines of discontinuity of the coefficients. The investigation is carried out with the aid of heat potentials using fundamental solutions constructed by Pogorzelski ([3,4,6–8]) and results of Gevrey ([1]).

Let us note that in the author' s paper ([9]) (cf. also ([10,11])) the existence of solutions of the principal boundary-value problems for a parabolic equation with discontinuous coefficients was proved by means of potentials, when the leading coefficient belonged to the class $C^{(1,\alpha)}$. This requirement was not dictated by the essence of the problems, but was a consequence of the application of Gevrey' s method ([1]). In the present paper the indicated restriction on the leading coefficient is removed, and the existence of the boundary-value problems is proved under the more natural assumption that the coefficients of the equation belong (in their domain of smoothness) to the class $C^{(\alpha)}$.

1. Consider in the (x, t)-plane the domains

$$S_T^{(i)} = \{(x,t);\ X_i(t) < x < X_{i+1}(t);\ 0 < t < T\}, \qquad i = 1, 2.$$

The functions $X_j(t)$ $(j = 1, 2, 3)$, determining the lateral boundaries of $S_T^{(i)}$, satisfy, in t, a Hölder condition with exponent $> 1/2$

$$|X_j(t) - X_j(t_1)| \leq K|t - t_1|^{\varkappa}, \qquad {}^1\!/_2 < \varkappa \leq 1; \tag{1}$$

where K and $\varkappa$ are constants.

The curves $x = X_j(t)$ have no common points. In the domain $S_T^{(i)}$ consider a parabolic equation of the form

$$L^{(i)}(u_i) \equiv a_i(x,t)\frac{\partial^2 u_i}{\partial x^2} + b_i(x,t)\frac{\partial u_i}{\partial x} + c_i(x,t)u_i - \frac{\partial u_i}{\partial t} = f_i(x,t) \tag{2}$$

and we shall seek a solution $u_i(x, t)$ of equation (2) satisfying the initial data

$$u_i(x, 0) = F_i(x), \qquad X_i(0) \leq x \leq X_{i+1}(0), \qquad i = 1, 2; \tag{3}$$

boundary conditions

$$\frac{\partial u_i}{\partial x}(X_j(t), t) + \lambda_i(t)u_i(X_j(t), t) = \varphi_i(t), \qquad 0 \leqslant t \leqslant T, \tag{4}$$

$$j = 1 \quad \text{for } i = 1, \qquad j = 3 \quad \text{for } i = 2$$

or

$$u_i(X_j(t), t) = \psi_i(t), \qquad 0 \leqslant t \leqslant T, \tag{5}$$

$$j = 1 \quad \text{for } i = 1, \qquad j = 3 \quad \text{for } i = 2;$$

conjugation conditions on the line of discontinuity

$$\alpha_1(t)\frac{\partial u_1(X_2(t), t)}{\partial x} - \alpha_2(t)\frac{\partial u_2(X_2(t), t)}{\partial x} = h(t), \qquad 0 \leqslant t \leqslant T; \tag{6}$$

$$u_1(X_2(t), t) - u_2(X_2(t), t) = r(t), \qquad 0 \leqslant t \leqslant T \tag{7}$$

and compatibility conditions

$$F_i'(X_j(0)) + \lambda_i(0)F_i(X_j(0)) = \varphi_i(0), \tag{8}$$

$$j = 1 \quad \text{for } i = 1, \qquad j = 3 \quad \text{for } i = 2 \quad \text{for (4)}$$

or

$$F_i(X_j(0)) = \psi_i(0), \tag{9}$$

$$j = 1 \quad \text{for } i = 1, \qquad j = 3 \quad \text{for } i = 2 \quad \text{for (5)}$$

and

$$\alpha_1(0)F_1'(X_2(0)) - \alpha_2(0)F_2'(X_2(0)) = h(0); \tag{10}$$

$$F_1(X_2(0)) - F_2(X_2(0)) = r(0). \tag{11}$$

We shall seek a solution $u_i(x,t)$ $(i = 1, 2)$ of problem (2)–(11), satisfying equation (2) inside the domain $S_T^{(i)}$ and continuous together with $\partial u_i/\partial x$ on $\overline{S}_T^{(i)}$. Everywhere in §1 the conditions I–IV are assumed to hold:

I. The operator $L^{(i)}$ in (2) is parabolic in $\overline{S}_T^{(i)}$.

II. In $\overline{S}_T^{(i)}$ the coefficients $a_i(x,t)$, $b_i(x,t)$, and $c_i(x,t)$ are continuous and satisfy a Hölder condition in x with a nonzero exponent. The coefficient $a_i(x,t)$ also satisfies a Hölder condition in t with a nonzero exponent. Let the rectangular domains

$$G_T^{(i)} = \{(x,t) : c_i < x < c_{i+1},\ 0 \leqslant t \leqslant T\} \quad (i = 1, 2)$$

be such that $\overline{S}_T^{(i)} \subset G_T^{(i)}$; then the coefficients a_i, b_i, and c_i will be regarded as extended to the domain $\overline{G}_T^{(i)}$ with preservation of properties I, II (see ([2])).

III. The functions $f_i(x,t)$ are continuous in $\overline{S}_T^{(i)}$ and satisfy a Hölder condition in x with a nonzero exponent.

IV. The initial function $F_i(x)$ is twice continuously differentiable, and its second derivative satisfies a Hölder condition in x with a nonzero exponent. When condition IV is fulfilled, it is clear that in problem (2)–(11) the nonhomogeneous initial conditions (3) may be regarded as replaced by homogeneous ones.

Theorem 1. *Let conditions (1), I—IV be fulfilled, and:*

a) the functions $\alpha_i(t)$, $\lambda_i(t)$, $\varphi_i(t)$ $(i = 1, 2)$, $h(t)$, occurring in the boundary conditions (4), (6), (8), (10), be continuous for $0 \leqslant t \leqslant T$; b) the functions $\psi_i(t)$ $(i = 1, 2)$, $r(t)$, occurring in conditions (5), (7), (9), (11), satisfy a Hölder condition in t with exponent $> 1/2$; c)

$$\alpha_1(t)\sqrt{a_2(X_3(t), t)} + \alpha_2(t)\sqrt{a_1(X_2(t), t)} \neq 0, \qquad 0 \leqslant t \leqslant T.$$

Then there exists a solution $u_i(x, t)$ of the boundary-value problem (2)—(11), continuous together with $\partial u_i/\partial x$ in the closed domain $\overline{S}_T^{(i)}$.

We outline the proof of Theorem 1. Under conditions I—II, Pogorzelski ([3,4]) proved the existence, in the bounded domain $S_T^{(i)}$, of a fundamental solution $\Gamma_i(x, t; \xi, \tau)$ $(i = 1, 2)$ for equation (2).

where

$$\Gamma_i(x, t; \xi, \tau) = \omega_i(x, t; \xi, \tau) + \mathfrak{w}_i(x, t; \xi, \tau),$$

$$\omega_i(x, t; \xi, \tau) = \frac{1}{\sqrt{t-\tau}} \exp\left\{-\frac{(x-\xi)^2}{4a_i(\xi, \tau)(t-\tau)}\right\}, \tag{12}$$

$$\mathfrak{w}_i(x, t; \xi, \tau) = \int_\tau^t d\eta \int_{c_i}^{c_{i+1}} \omega_i(x, t; y, \eta)\Phi_i(y, \eta; \xi, \tau)\, dy.$$

The function $\Phi_i(x, t; \xi, \tau)$ satisfies the estimate

$$|\Phi_i(x, t; \xi, \tau)| \leq \frac{C}{(t-\tau)^\mu}\, \frac{1}{|x-\xi|^{3-2\mu-\alpha_1}}, \tag{13}$$

$\alpha_1 = \min(\alpha, 2\beta)$ (α and β are the exponents in the Hölder conditions for $u_i(x, t)$ with respect to x and t, respectively); μ is any positive number satisfying the inequality $1 - \alpha_1/2 < \mu < 1$.

Introducing the solutions of equation (2) that satisfy the zero initial conditions

$$Z_i(x, t) = -\frac{1}{2\sqrt{\pi}} \iint_{S_t^{(i)}} \frac{f_i(\xi, \tau)}{\sqrt{a_i(\xi, \tau)}}\, \Gamma_i(x, t; \xi, \tau)\, d\xi\, d\tau \quad (i = 1, 2),$$

with the aid of Pogorzelski' s estimate (13) and Gevrey' s results ([1]), problem (2) —(11) is reduced to an analogous problem for a homogeneous equation. Seeking the solution of this problem in the form of simple-layer potentials

$$u_i(x,t) = \sum_{j=i}^{i+1} \int_0^t \Gamma_i(x,t;X_j(\tau),\tau)\Psi_{j+i-1}(\tau)\,d\tau \quad (i=1,2),$$

and using the boundary conditions and the jump formula for the derivative with respect to x of a simple-layer potential, one can reduce the problem to the solution of a system of singular Volterra integral equations of the first and second kinds for the four unknown functions $\Psi_i(t)$ $(i=1,2,3,4)$. The equations of the first kind corresponding to the boundary conditions (5) and (7), by a known device (see ([1])), reduce to equations of the second kind. The finally obtained system of four Volterra integral equations of the second kind has kernels with weak singularities (which is shown with the aid of Gevrey' s results ([1]) and estimate (13)) and is solved by the usual method of successive approximations.

Remark 1. Theorem 1 remains valid for any finite number of discontinuity lines on which conjugation conditions of type (6) are prescribed.

Remark 2. If $a_i(x,t)$ has $\partial a_i/\partial x$ and $\partial a_i/\partial t$ in $C^{(\alpha)}$, and $X_j(t)$ has a derivative satisfying (1), then an analogous result was obtained by A. A. Samarskii ([5]).

2. Let us now consider problem (2)–(11) in the case when one or both domains $S_T^{(i)}$ $(i=1,2)$ become infinite in x. Let, for example,

$$\overline{S}_T^{(2)} = \{(x,t);\ X_2(t) < x < +\infty,\ 0 < t < T\}.$$

Then, if in (4), (5), (8), (9) we put $i=1$, the following holds.

Theorem 2. Suppose that conditions (1), I, and

V. The coefficients a_i, b_i, and c_i satisfy condition II and, in addition, are uniformly continuous with respect to t and bounded in $\overline{S}_T^{(2)}$.

VI. The function $f_1(x,t)$ satisfies condition III, and $f_2(x,t)$ is continuous in $\overline{S}_T^{(2)}$, and with respect to x satisfies a Hölder condition of the form

$$|f_2(x,t) - f_2(x_1,t)| \le D|x-x_1|^\gamma e^{a|x|}, \quad |x_1| \le |x|,$$

$$0 < \gamma \le 1.$$

and the inequality

$$|f_2(x,t)| \leqslant F_2 e^{a|x|},$$

where $F_2, a > 0$ are constants.

VII. The initial function $F_1(x)$ satisfies IV, and $F_2(x)$ is twice continuously differentiable and

$$|F_2''(x) - F_2''(x_1)| \leqslant \varepsilon |x - x_1|^\delta e^{b|x|}, \qquad |x_1| \leqslant |x|, \qquad 0 < \delta \leqslant 1,$$

$$|F_2^{(k)}(x)| \leqslant \varepsilon e^{b|x|} \quad (k = 0, 1, 2), \qquad b > 0,$$

where ε, b, δ are constants.

VIII. The functions $\alpha_i(t)$ $(i = 1, 2)$, $h(t)$, $\lambda(t)$, $\varphi_1(t)$, $\psi_1(t)$, $r(t)$, entering into the boundary conditions, satisfy conditions a), b), c) of Theorem 1.

Then there exists a solution $u_i(x, t)$ of the mixed problem (2)–(11) with an infinite domain of initial values, continuous together with $\partial u_i / \partial x$ in the closed domain $\overline{S_T^{(i)}}$.

The proof of Theorem 2 is carried out according to the scheme of item 1, using, in the domain $\overline{S_T^{(2)}}$, unbounded with respect to x, the fundamental solution $\Gamma_2(x, t; \xi, \tau)$ (12) (where $c_2 = -\infty, c_3 = +\infty$), constructed by Pogorzelski for unbounded domains [7,8], and the estimates (13) for $|x| \leqslant X$ and

$$|\Phi_2(x, t; \xi, \tau)| \leqslant \frac{C(t-\tau)^{\mu_1}}{|x-\xi|^{1+2\mu_1}} e^{-k|x-\xi|}$$

for $|x| \geqslant X$, where μ_1 and k are arbitrary positive constants [7].

Remark 3. Instead of one domain unbounded in x, one may, of course, consider two unbounded domains, between which a finite number of lines of discontinuity of the coefficients (2) is enclosed. Then we obtain an existence theorem for the solution of the Cauchy problem for the parabolic equation (2) of parabolic type with discontinuous coefficients.

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Note: Figure translations are in progress. See original paper for figures.