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Abstract

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ESTIMATES OF DERIVATIVES OF SOLUTIONS OF ELLIPTIC EQUATIONS NEAR THE BOUNDARY

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In the present paper the first boundary-value problem is considered for the elliptic equation:

$$Lv \equiv \sum_{k_1 k_2 + \dots + k_n = 2m} A^{(k_1 \dots k_n)}(x_1 \dots x_n) \frac{\partial^{2m} u}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} + Tu = f, \quad (1)$$

where Tu denotes a linear differential operator of order $< 2m$. All coefficients are assumed sufficiently smooth. The first boundary-value problem for such an equation may be posed as follows: find a generalized solution in a bounded domain D , belonging to the space $\overset{\circ}{W}_2^m$. This space is the closure of the manifold of smooth functions that vanish in a boundary strip, in the metric W_2^m . Normal solvability of such a problem follows, for example, from the work of M. I. Vishik ⁽¹⁾. In the case of a sufficiently smooth boundary the solution is differentiable up to the boundary and the estimate ⁽²⁾

$$\|u\|_{W_2^{2m}} \leq \|f\|_{L_2} \|u\|_{L_2}. \quad (2)$$

is established.

It is known that for the equation $\Delta u = f$ in the plane, in the presence of angular points of the boundary, estimate (2) may fail to hold ⁽⁶⁾. In the present paper the question of the smoothness of the solution up to the boundary and of estimates of derivatives is studied for certain classes of domains whose boundary is not differentiable. The analogous question for the Laplace equation in the plane for domains with angular points was completely investigated in ⁽³⁾.

Lemma 1. *Let $n = 2$, and let a point A , belonging to the boundary of the domain D , be such that it is accessible from outside D by a Jordan curve. Then for any function $v \in \overset{\circ}{W}_2^m$ the inequality*

$$\iint_{S(A)} \frac{v^2}{r^{2m}} dS \leq C \iint_{S(A)} \sum (D_m v)^2 dS, \quad (3)$$

holds, where r is the distance to the point A , and the integration is carried out over a disk with center at A of sufficiently small radius ρ . The constants C and

ρ do not depend on v . The summation on the right-hand side of (3) is over all derivatives of order m . We assume that $v \equiv 0$ outside D .

This lemma may be proved as follows. Draw a circle S with center at the point A of such radius ρ that it intersects the curve along which the point A is accessible from outside D . If Q is an arbitrary point on this circle, then

$$v(Q) = v(Q_0) + \int_{Q_0}^Q \frac{\partial v}{\partial \varphi} ds,$$

where Q_0 is the point of intersection of S and the curve along which the point A is accessible from outside D .

Since $v(Q_0) = 0$, applying the Cauchy–Bunyakovsky inequality, we obtain:

$$\int_S v^2 d\varphi \leq c_1 \int_S \text{grad}^2 v d\varphi.$$

If $v \in \overset{\circ}{W}_2^m$, then this inequality can

apply also to the successive derivatives of v up to order $(m - 1)$, whence it follows that $\int_S v^2 d\varphi \leq C_1 \int_S \sum (D^m v)^2 d\varphi$. Integrating this inequality over all S of radii less than ρ , we obtain (3).

If $n \geq 2$, then in an analogous way one can obtain Lemma 1':

Lemma 1'. *If $n \geq 2$ and the point A , belonging to the boundary of the domain D , is such that the angular measure of the intersection of the surface of a sphere with center at the point A and of the complement of D exceeds some number $\alpha > 0$ for all sufficiently small spheres, then the inequality*

$$\iint_{S(A)} \frac{v^2}{r^{2m}} dS \leq \frac{C(\omega_n - \alpha)}{\alpha} \iint_{S(A)} \sum (D^m v)^2 dS \quad (3')$$

is valid for $v \in \overset{\circ}{W}_2^m$.

Here ω_n is the surface area of the unit sphere in n -dimensional space.

Inequalities (3) and (3') give some estimates of the decrease of the solution of the homogeneous first boundary value problem for equation (1) near "regular" boundary points ("regular" in the sense that the conditions of Lemmas 1 and 1', respectively, are satisfied).

In the following theorems some inequalities will be established that estimate, on the average, the rate of decrease of the solution and of its derivatives of order m in a neighborhood of boundary points and strengthen (3) and (3').

Theorem 1. *If the conditions of Lemma 1 are satisfied, then*

$$\iint_{S(A)} \frac{u^2}{r^{2m+2\beta}} dS \leq C_1 \iint_{S(A)} \frac{\sum (D^m u)^2}{r^{2\beta}} dS \leq C_2 \iint_{S(A)} f^2 r^{2m-2\beta} dS, \quad (4)$$

where u is the solution of the homogeneous first boundary value problem for equation (1), and β is any number less than a certain constant β_0 , which depends only on the values of the coefficients of the highest part of equation (1).

In the case $n \geq 2$, Theorem 1' holds:

Theorem 1'. *If the conditions of Lemma 1' are satisfied, then*

$$\iint_{S(A)} \frac{u^2}{r^{2m+2\beta}} dS \leq C_1 \iint_{S(A)} \frac{\sum (D^m u)^2}{r^{2\beta}} dS \leq C_2 \iint_{S(A)} f^2 r^{2m-2\beta} dS \quad (4')$$

for any solution of the homogeneous first boundary value problem for equation (1).

Here β is any number less than $l(\omega_n - \alpha)/\alpha$, where l is a constant depending on the values of the coefficients of the principal part of equation (1), and α is the constant entering into the formulation of Lemma 1'.

These theorems are proved as follows. In equation (1) the substitution $u = vr^\beta$ is made. For the function v a new equation is obtained, whose coefficients at the derivative of order $2m - k$ have a singularity of order $\frac{1}{r^k}$. It can be proved that the new equation has a solution in the space W_2^m . This can be done in the same way as was done in the work of M. I. Vishik [1], with the difference that the lower coefficients in our case have singularities and do not constitute a completely continuous operator. However, they constitute a bounded operator whose norm, when the conditions of Theorems 1 and 1' in the domain D are satisfied, is less than 1, and the arguments given in [1] lead to the equation $u + Bu + iCu = \varphi$, where $\|B\| < 1$, $C^* = C$, and this equation is solvable. Thus, $v \in W_2^m$, and hence inequalities (3) and (3') are valid for v . Taking into account that $u = vr^\beta$, we obtain (4) and (4').

We note that if $Lu \equiv \Delta u$ and $n = 2$, then $\beta_0 = \frac{1}{2}$. Using Theorems 1 and 1', one can establish estimate (2) for certain domains.

Theorem 2. *If the boundary of the domain D contains a finite number of points in neighborhoods of which, in local coordinates, D is a cone whose opening is smaller than a certain number α_0 , depending only on the coefficients of equation (1), then inequality (2) holds.*

It is clear that it suffices to establish inequality (2) in a neighborhood of each conical point. Let A be one of them and let G_n be the intersection of D and the layer

$$\frac{1}{2^{n+1}} \leq r \leq \frac{1}{2^n}.$$

It is easy to obtain the estimate:

$$\|u\|_{W_2^{2m}}^{G_n} \leq C \left[2^{2m} \|u\|_{L_2}^{G_n+G_{n-1}+G_{n+1}} + \|f\|_{L_2}^{G_n+G_{n-1}+G_{n+1}} \right]. \quad (5)$$

This inequality for G_2 follows from (2'), and for G_n it is obtained after a similarity transformation of G_n into G_2 .

Rewrite (5) as follows:

$$\|u\|_{W_2^{2m}}^{G_n} \leq C \iint_{G_{n-1}+G_n+G_{n+1}} f^2 dS + C \iint_{G_{n-1}+G_n+G_{n+1}} \frac{u^2}{r^{4m}} dS. \quad (6)$$

If α is so small that Theorem 1' gives the estimate

$$\iint \frac{u^2}{r^{4m}} dS \leq \iint_{S(A)} f^2 dS,$$

then, summing (6) over all n , we obtain (2).

We note that the result of Theorem (2) remains valid if u is the solution of the inhomogeneous problem

$$\partial^k u / \partial n^k = \varphi_k, \quad k = 0, 1, \dots, n-1, \quad \varphi_k \in C^{2m-k-1/2+\varepsilon},$$

and the compatibility conditions at A are satisfied. In the case $n = 2$ these conditions are written out in (4, 5); this can similarly be done in the general case.

The following theorem is an estimate of Schauder type:

Theorem 3. If the boundary D in a neighborhood of the point A , in curvilinear local coordinates, is a cone whose opening is ω , and $\omega \neq k_i$, where k_i is a countable sequence of numbers depending only on the values of the coefficients of the equation at the point A , then

$$\|u\|_{C^{r-\varepsilon}} \leq \|f\|_{C^{r-2m}} + \sum_{j=0}^{m-1} \|\varphi_j\|_{C^{r-j}}.$$

Here $r = \left[\frac{K}{\omega} \right]$, and K is a constant depending only on equation (1).

The sequence k_i is defined as follows. We take all infinitely differentiable functions w equal to zero on the lateral surface of the cone of opening ω . If Lw , $\frac{\partial}{\partial x} Lw$, $\frac{\partial^q}{\partial x_1^{q_1} \dots \partial x_n^{q_n}} Lw$ can, by a suitable choice of w , be made equal to any prescribed numbers, then $\omega \neq k_i$. The number of such ω for which this cannot be done turns out to be countable. If $Lu \equiv \Delta u$ and $n = 2$, then $k_i = \pi/i$.

We note that if in the formulation of Theorem 3 one requires $f \in C^{r+\varepsilon}$, then it may be asserted that $u \in C^{2m+r+\varepsilon}$, while for $\varepsilon = 0$ this is false.

The proof of Theorem 3 is analogous to the proof of Theorem 2.

Theorems of the type of Theorems 2 and 3 hold for dihedral angles and, in general, for domains that are simplexes.

A number of corollaries follow from Theorem 3.

Corollary 1. If $f \in C^\infty$, and the opening of the cone $\omega \neq k_i$, then the solution is differentiable up to the boundary $\left[\frac{K}{\omega}\right]$ times.

This corollary means that the smoothness of the solution improves as the angle decreases.

Corollary 2. If in a neighborhood of a boundary point the domain is a power cone with zero angle and $f \equiv 0$ in this neighborhood, then the solution belongs to C^∞ . If, however, $f \in C^\infty$ and the compatibility conditions are satisfied, then also $u \in C^\infty$.

Let us also consider the case when the domain D in a neighborhood of the origin has the form:

$$x_n \geq 0; \quad \text{for } x_n < 0 \quad x_n^2 \geq \psi \left(\sum_{i=1}^{n-1} x_i^2 \right), \quad \psi(0) = 0, \quad \psi(t) \geq 0.$$

If $2m < n$, then the solution may be discontinuous at the origin. But the following theorem turns out to be true:

Theorem 4. If $|\psi(t)| \leq t^{n-2m+k+\varepsilon}$, then the solution of the first boundary-value problem is k times continuously differentiable in the domains: $x_n \geq 0$; $x_n^2 \geq c \sum_{i=1}^{n-1} x_i^2$ for $x_n < 0$, with an arbitrary constant c .

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Note: Figure translations are in progress. See original paper for figures.

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