Mathematics

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1962

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Abstract

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Fixed Singular Points of Generalized Analytic Functions

Consider the generalized Cauchy–Riemann equation

$$\frac{\partial w}{\partial \bar{z}} + A(z)w + B(z)\bar{w} = 0, \tag{1}$$

whose coefficients satisfy the following conditions: there exist analytic functions $\varphi(z)$ and $\psi(z)$ such that

$$A_1 = \frac{A}{\varphi} \in L_p(G), \qquad B_1 \equiv \frac{B}{\psi} \in L_p(G),$$

where p is some number > 2, and G is a bounded domain in the plane $z = x + iy$. It is known (^{1}) that the solutions of equation (1), if they exist, have the form

$$w(z) = \Phi(z)\exp[\varphi(z)\omega(z) + \varphi(z)\chi(z)], \tag{2}$$

where

$$\omega = \frac{1}{\pi}\iint_G \frac{A_1(\zeta)\,d\xi\,d\eta}{\zeta - z} \equiv T(A_1), \qquad \chi = T\left(B_1\frac{\bar{w}}{w}\right).$$

These functions belong to the class $C_\alpha(E)$, $\alpha = \dfrac{p-2}{p}$; E denotes the whole plane.

The functions φ and ψ, which are called analytic regularizers, may have inside G only a discrete set of isolated singular points—poles and essential singular points. We shall call these points isolated singular points of regular type of equation (1). In the presence of such points, as is seen from formula (2), the solutions of equation (1) have isolated points of two types: 1) poles and essential singular points of the analytic regularizers φ and ψ, which we shall call fixed singular points, and 2) poles and essential singular points of the analytic function Φ, not coinciding with the fixed singular points. We shall call these singular points movable.

Let us now pose the following problem:

Suppose a certain analytic function $\Phi(z)$ is given in the domain G. It is required to construct a solution of equation (1) having the form (2).

As is known (^{1},^{2}), this problem always has a solution if one can take $\varphi = \psi = 1$ as the regularizers. Below we shall prove that the problem admits a solution also in the general case, when φ and ψ are arbitrary analytic functions in G, which may have any number of poles and essential singular points.

First consider an equation of the form

$$\frac{\partial w}{\partial \bar{z}} + f(z, w) = 0. \tag{3}$$

Assume that there exist a function $w(z)$ satisfying equation (3), and an analytic function $\varphi(z)$ such that

$$\frac{f(z, w(z))}{\varphi(z)} \in L_1(G).$$

Then the function $w(z)$ is represented in the form (1)

$$w(z) = \Phi(z) + \varphi(z)\, T\left(\frac{f}{\varphi}\right).$$

Introducing the new function $u = \dfrac{w - \Phi}{\varphi}$, we obtain

$$u = T_0(u) \equiv \iint_D \frac{f_0(\zeta, u(\zeta))}{\zeta - z}\, d\xi\, d\eta, \tag{4}$$

where

$$f_0(\zeta, u) \equiv \frac{f(\zeta, \Phi(\zeta) + \varphi(\zeta)u(\zeta))}{\pi\varphi(\zeta)}.$$

It is now not difficult to see that if u satisfies equation (4), and $f_0(\zeta, u(\zeta)) \in L_1(G)$, then the function $w = \Phi + \varphi u$ will be a solution of equation (3).

In what follows we shall consider this problem in a somewhat narrower formulation. We first prove the following lemma.

Lemma. Let φ and Φ be given analytic functions in G. Suppose that for every function u of the class $C_\alpha(D)$, $\alpha = \dfrac{p-2}{p}$, $p > 2$ (1), the condition

$$\left(\iint_D |f_0(\zeta, u(\zeta))|^p \, d\xi \, d\eta\right)^{1/p} \equiv L_p(f_0) \leq M, \qquad p > 2, \tag{5}$$

is fulfilled, where the constant M does not depend on the choice of the function u. Then there exists a function of the class $D_{1,p}(G)$, $p > 2$, satisfying equation (4).

Proof. Consider the set K_M of functions of the form

$$u = \frac{1}{\pi} \iint_G \frac{\mu(\zeta) \, d\xi \, d\eta}{\zeta - z} \equiv T(\mu), \qquad L_p(\mu) \leq M, \qquad p < 2.$$

This set is uniformly bounded and equicontinuous (1):

$$|u| \leq M_0 M, \qquad |u(z_1) - u(z_2)| \leq M_0 M \, |z_1 - z_2|^{\frac{p-2}{p}}.$$

Therefore, from every infinite sequence of elements of the set K_M one can extract a subsequence $T(\mu_n)$ which converges uniformly inside G. Since $L_p(\mu_n) \leq M$, from the sequence μ_n one can extract a subsequence μ_{n_k} which converges weakly in L_p to some function μ_0 of class L_p, with $L_p(\mu_0) \leq M$. Taking into account that the function $(\zeta - z)^{-1}$ belongs to every class $L_{p'}(G)$, $p' < 2$, we have: $T(\mu_{n_k}) \to T(\mu_0)$, and the convergence is uniform inside G.

Thus the compactness of K_M is proved. Moreover, obviously, K_M is a convex set. Since K_M is a subset of the Banach space $C_\alpha(D)$, it follows, by Schauder' s principle, that under the mappings $T_0(u)$ there exists in K_M a fixed element, i.e., equation (4) has a solution representable in the form $u = T(\mu)$, $L_p(\mu) \leq M$. The lemma is proved.

Equation (4) may, generally speaking, have many solutions. We shall indicate one sufficient condition ensuring uniqueness of the solution.

Suppose that for any two elements of the set K_M the condition

$$f_1(z) \equiv \frac{f_0(z, w_1(z)) - f_0(z, w_2)}{w_1 - w_2} \in L_p(G), \qquad p > 2 \tag{6}$$

is fulfilled.

Then equation (4) admits a unique solution of class $D_{1,p}$. Indeed, if there are two solutions w_1 and w_2, then the difference $w = w_1 - w_2$ will be a solution of the homogeneous Fredholm integral equation

$$w - \iint_G \frac{f_1(\zeta) w(\zeta) \, d\xi \, d\eta}{\zeta - z} = 0.$$

But this equation has only the trivial solution $w \equiv 0$ (1).

Thus, if for the prescribed analytic functions Φ and φ conditions (5) are satisfied, then equation (3) has a solution of the form: $w = \Phi + \varphi u$, where u is a solution of equation (4).

Consider now the equation

$$\frac{\partial w}{\partial \bar{z}} + f(z, w)w = 0. \tag{7}$$

Let, as before, φ and Φ be certain prescribed analytic functions. Let

$$f_2(z, u) = \frac{f\left(z, e^{\Phi(z)+\varphi(z)u(z)}\right)}{\varphi(z)}.$$

If for every function $u(z)$ of the class $C_\alpha(D)$, $\alpha = \frac{p-2}{p}$, the condition

$$L_p(f_2) < M, \qquad p > 2, \tag{8}$$

is satisfied, then equation (7) has a solution of the form

$$w(z) = \Phi(z)e^{\varphi(z)\omega(z)}, \tag{9}$$

where ω is a solution of the equation

$$\omega(z) - \iint_G \frac{f_2(\zeta, \omega(\zeta))}{\zeta - z}\, d\xi\, d\eta = 0, \tag{10}$$

which, by virtue of the lemma proved above, admits at least one solution of class $D_{1,p}(G)$, $p > 2$.

Let us now return to equation (1). Let the analytic functions φ and ψ be regularizers of the coefficients A and B, respectively. First make the substitution

$$w_* = we^{-\varphi(z)T\left(\frac{A}{\varphi}\right)}. \tag{11}$$

Then equation (1) is reduced to the form

$$\frac{\partial w_*}{\partial \bar{z}} + B_* \bar{w}_* = 0, \tag{12}$$

where

$$B_* = Be^{-2i\,\mathrm{Im}\left(\varphi T\left(\frac{A}{\varphi}\right)\right)}.$$

Obviously, B_* has the same regularizer φ as B. Write equation (12) in the form

$$\frac{\partial w_*}{\partial \bar{z}} + f(z, w)w_* = 0, \qquad f = B_* \frac{\bar{w}_*}{w_*}. \tag{13}$$

As is not difficult to see, condition (8) in the present case is satisfied for arbitrary functions Φ and u. Consequently, equation (14) has a solution of the form (9). It is not difficult to see that in the present case condition (6), ensuring the uniqueness of the solution of equation (10), is satisfied. Substituting (9) into (11), we obtain formula (2), the proof of which was our principal aim.

In conclusion we note that the special case where $\varphi = \psi = z^{-1}$ was studied earlier by another method in paper (3).

Received
4 IV 1962

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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