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Abstract

Full Text

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ASYMPTOTIC METHOD FOR CERTAIN DIFFERENTIAL EQUATIONS WITH OSCILLATING COEFFICIENTS

(Presented by Academician N. N. Bogolyubov, 15 XI 1961)

MATHEMATICS

1. We shall consider differential equations of the form

$$\frac{d\psi}{dt} = H(t)\psi(t), \quad \psi(0) = \psi_0, \quad (1)$$

where $H(t)$, for each t , is an operator in the Hilbert space \mathfrak{H} , and $\psi(t)$ is a function with values in \mathfrak{H} .

Suppose that $H(t)$ depends in a special way on a small parameter ε , namely, has the form

$$H(t, \varepsilon) = H_0 + \sum_{k=1}^{\infty} \sum_{m=-\infty}^{\infty} \varepsilon^k H_{km}(\tau) e^{im\omega t} \quad (\tau = \varepsilon t). \quad (2)$$

Here it is assumed that the series (2) converges asymptotically, and, for simplicity, it is assumed that for each k only a finite number of the operators $H_{km}(\tau)$ are different from zero. All these operators are assumed to be differentiable sufficiently many times with respect to τ .

Below we shall indicate a method for constructing asymptotic approximations to the solution of equation (1). The proposed process depends essentially on the relation of the number ω to the spectrum Λ of the operator H_0 .

In the case when the expansion (2) contains no oscillating terms, $H_{km}(\tau) = 0$ ($m \neq 0$), the proposed method is well studied. In the finite-dimensional case it was considered by Birkhoff, Tryjitzinsky, and others (see ¹), and later in the works of V. S. Pugachev, I. Z. Shtokalo, S. F. Feshchenko ²⁻⁴. It was extended to equations in an infinite-dimensional space by S. G. Krein and the author ⁵⁻⁷.

2. Let us first consider formally the algebraic side of the process. Suppose that the spectrum Λ of the operator H_0 decomposes into closed nonintersecting parts

$$\Lambda = \Lambda_0 + \Lambda_1 + \dots + \Lambda_p,$$

where only one of them, Λ_0 , may contain an infinitely distant point.

Denote by

$$P_k = -\frac{1}{2\pi i} \oint_{\Gamma_k} (H_0 - \mu I)^{-1} d\mu \quad (k = 1, 2, \dots, p),$$

$$P_0 = I - \sum_{k=1}^p P_k$$

the operators projecting onto invariant subspaces \mathfrak{H}_k of the operator H_0 , in each of which the spectrum of the operator H_0 coincides with the corresponding set Λ_k . Here Γ_k is a simple smooth closed contour surrounding Λ_k and separating it from the remaining part of the spectrum.

The operator $U(t, \varepsilon)$, expressing the solution of equation (1) in terms of its initial value,

$$\psi(t, \varepsilon) = U(t, \varepsilon)\psi_0,$$

will be sought in the form

$$U(t, \varepsilon) = \sum_{s=0}^p U_s(\tau, t, \varepsilon) Y_s(t, \varepsilon), \quad (3)$$

where the operators Y_s act in the subspaces \mathfrak{S}_s : $Y_s = P_s Y_s = Y_s P_s$ and satisfy in them the differential equations

$$\frac{dY_s}{dt} = \Omega_s(\tau, \varepsilon) Y_s \quad (s = 0, 1, \dots, p). \quad (4)$$

It is assumed that the coefficients from (3) and (4) are representable in the form

$$\Omega_s(\tau, \varepsilon) = H_0 P_s + \sum_{r=1}^{\infty} \varepsilon^r \Omega_{sr}(\tau); \quad U_s(\tau, t, \varepsilon) = \sum_{r=0}^{\infty} \varepsilon^r \sum_{m=-\infty}^{\infty} e^{im\omega t} U_{sr}^m(\tau). \quad (5)$$

It is necessary to determine the operators U_{sr}^m and Ω_{sr} in such a way that, at least formally, the operator $U(t, \varepsilon)$ satisfy the equation

$$\frac{dU(t, \varepsilon)}{dt} = H(t, \varepsilon)U(t, \varepsilon), \quad U(0, \varepsilon) = I. \quad (6)$$

In order to do this, substitute (3) into (6) and, using relations (2), (4), (5), equate the coefficients at the various expressions of the form $\varepsilon^r Y_s e^{im\omega t}$. This yields the recurrent system of relations

$$\left\{ \sum_{r_1=0}^r U_{sr-r_1}^m(\tau) \Omega_{sr_1}(\tau) + im\omega U_{sr}^m(\tau) + U_{sr-1}^{m'}(\tau) \right\} P_s$$

$$= \sum_{m_1=-\infty}^{\infty} \sum_{r_1=0}^r H_{r-r_1}^{m_1}(\tau) U_{sr_1}^{m-m_1}(\tau) P_s$$

$$(r = 0, 1, \dots; s = 0, 1, \dots, p; -\infty < m < \infty). \quad (7)$$

Moreover, the operator $U(t, \varepsilon)$ must satisfy the condition $U(0, \varepsilon) = I$. For $\varepsilon = 0$ it is sufficient for this that the relations

$$Y_s(0, 0) = P_s, \quad U_{s0}^m(0) = P_s \delta_{m0} \quad (8)$$

hold.

For $r = 0$, from (7) the equality follows

$$(H_0 - im\omega)U_{s0}^m P_s - U_{s0}^m H_0 P_s, \quad (9)$$

which is satisfied identically if one sets $m = 0$, $U_{s0}^0 = P_s$. For $m \neq 0$ we multiply (9) on the left by P_σ :

$$(H_0 - im\omega)P_\sigma U_{s0}^m P_s - P_\sigma U_{s0}^m P_s H_0 = 0. \quad (10)$$

If the condition

$$\Lambda_\sigma - im\omega \cap \Lambda_s = 0, \quad (11)$$

is fulfilled, then it follows from (10) that $P_\sigma U_{s0}^m P_s = 0$. If condition (11), for some m and $\sigma = \sigma(m, s)$, is violated, then such an equality would already lead to a contradiction.

Below we shall consider only the case when condition (11) can be violated only for parts of the spectrum of the operator H_0 consisting of a single point, assuming thereby that the corresponding invariant subspaces of H_0 are its eigenspaces. In other words, $\sigma = \sigma(m, s)$ is possible only under the condition $\Lambda_\sigma = \lambda_\sigma$, $\Lambda_s = \lambda_s$; $H_0 P_\sigma = \lambda_\sigma P_\sigma$; $H_0 P_s = \lambda_s P_s$. In this case it turns out that $\lambda_\sigma - im\omega = \lambda_s$, and (10) becomes an identity, while the operator $P_{\sigma(m,s)} U_{s0}^m(\tau) P_s$ remains undetermined.

For $r = 1$ relation (7), multiplying it on the left by P_σ , can be written in the form

$$(H_0 - im\omega)P_\sigma U_{s1}^m P_s - P_\sigma U_{s1}^m P_s H_0 =$$

$$= P_\sigma U_{s_0}^m \Omega_{s_1} P_s + P_\sigma U_{s_0}^{m'} P_s - \sum_{m_1} P_\sigma H_1^{m_1} U_{s_0}^{m-m_1} P_s. \quad (12)$$

For $m = 0$ and $\sigma = s$ we use $P_s U_{s_1}^0 P_s = 0$ and obtain the equality

$$\begin{aligned} \Omega_{s_1}(\tau) &= P_s \Omega_{s_1}(\tau) P_s = \sum_{m_1} P_s H_1^{m_1}(\tau) U_{s_0}^{-m_1}(\tau) P_s = \\ &= P_s H_1^0(\tau) P_s + \sum_{m_1 \neq 0} P_s H_1^{m_1} U_{s_0}^{-m_1} P_s, \end{aligned} \quad (13)$$

where in the last sum there enter only the terms of zero order with respect to ε that have remained undetermined.

For $m = 0$ and $\sigma \neq s$ one obtains the equation

$$H_0 P_\sigma (P_\sigma U_{s_1}^0 P_s) - (P_\sigma U_{s_1}^0 P_s) H_0 P_s = - \sum_{m_1} P_\sigma H_1^{m_1} U_{s_0}^{-m_1} P_s, \quad (14)$$

from which the operator $P_\sigma U_{s_1}^0 P_s$ can be found, for a known right-hand side, as indicated below. An equation of the same type will also be obtained in the case when $m \neq 0$, $\sigma = \sigma(m, s)$. If, however, $\sigma = \sigma(m, s)$, then, as above, the left-hand side of (12) under the assumptions we have adopted vanishes and the relation

$$P_\sigma U_{s_0}^m(\tau) P_s \Omega_{s_1}(\tau) + P_\sigma U_{s_0}^{m'}(\tau) P_s - \sum_{m_1} P_\sigma H_1^{m_1}(\tau) U_{s_0}^{m-m_1}(\tau) P_s = 0$$

is obtained, which together with (13) gives an equation for the operator $X_{\sigma s}^m = P_\sigma U_{s_0}^m P_s$.

For fixed s and all m and $\sigma(m, s)$, these equations constitute a system of Riccati-type equations with respect to the as yet undetermined zero-order terms (in particular, under certain conditions, a single equation). These equations must be solved under the conditions $X_{\sigma s}^m(0) = 0$, which follow from (8). Generally speaking, it then turns out that $X_{\sigma s}^m(\tau) \neq 0$ for $\tau \neq 0$, and thus among the terms of zero order with respect to ε there are nonzero oscillating terms.

Equations of type (14), in the case when the operators $H_0 P_s$ and $H_0 P_\sigma$ are bounded, were considered in (6). From the formula given there it follows that

$$P_\sigma U_{s_1}^m(\tau) P_s = \frac{1}{4\pi^2} \oint_{\Gamma_\sigma} \oint_{\Gamma_s} \frac{(H_0 - im\omega - \lambda) \sum_{m_1} P_\sigma H_1^{m_1} U_{s_0}^{m-m_1} P_s (H_0 - \mu)^{-1} d\lambda d\mu}{\lambda - \mu - i\omega}. \quad (15)$$

If one of the sets $\Lambda_\sigma, \Lambda_s$ contains the point at infinity, then, as indicated in (7), the equation must be reduced to a convenient form by a fractional-linear transformation.

Thus, all first-order terms with respect to ε will be determined, except for the operators $P_\sigma U_{s1}^m P_s$ for $\sigma = \sigma(m, s)$, which are determined at the next step of the calculations.

All calculations at the second and subsequent steps are carried out in an entirely analogous way, and even more simply, since the differential equation for the operators $P_{\sigma(m,s)} U_{sr}^m P_s$ for $r \geq 1$ turns out to be linear, and not a Riccati equation.

3. Let us consider the second-order equation

$$\frac{d^2\psi}{dt^2} + A(t, \varepsilon) \frac{d\psi}{dt} + B(t, \varepsilon)\psi = 0. \quad (16)$$

By means of the usual device it can be reduced to a system of first-order equations

$$\frac{d\psi}{dt} = \psi_1; \quad \frac{d\psi_1}{dt} = -B\psi - A\psi_1$$

or, in other words, to a single equation of the form (1) in the space $\mathfrak{H} \oplus \mathfrak{H}$. Thus, the method described above can be applied to equation (16). An interesting example of such equations is furnished by equations of the form

$$\frac{d^2\psi}{dt^2} + \varepsilon [A_0(\tau) + A_1(\tau) \cos \omega t] \frac{d\psi}{dt} + \{B_0 + \varepsilon [B_{10}(\tau) + B_{11}(\tau) \cos \omega t]\} \psi = 0.$$

In the one-dimensional case, for $A_0 = A_1 = B_{10} = 0$, $B_{11} = \text{const}$, this equation becomes Mathieu's equation. If $\omega = 2\sqrt{B_0}$, then condition (11) is precisely violated.

4. The justification of the asymptotic convergence of the process under consideration does not differ from that given in (6,7). It is necessary to assume that the operators $H_0, H_{km}(\tau)$ satisfy the conditions ensuring the existence and well-posedness of solutions of the differential equations (4). For example, the operator H_0 must satisfy the well-known Hille-Yosida conditions, and the operators H_{km} must be bounded. In this case the space \mathfrak{H} may also be a Banach space.

If the operator H_0 is self-adjoint and positive definite, then one can consider the more general case, studied in (9), when the operators H_{km} have fractional order relative to H_0 .

If any of the above conditions are fulfilled, then, defining the operators

$$U_{sr}^m(\tau), \quad \Omega_{sr}(\tau) \quad (r \leq k), \quad \Omega_s^{(k)} = H_0 P_s + \sum_{r=1}^k \Omega_{sr}(\tau) \varepsilon^r$$

and finding the operators Y_s^k from the equations

$$\frac{dY_s^{(k)}}{dt} = \Omega_s^{(k)}(\tau, \varepsilon) Y_s^{(k)},$$

we construct the operator

$$V^{(k)}(t, \varepsilon) = \sum_{s=0}^p \sum_m \sum_{r=0}^k \varepsilon^r U_{sr}^m(\tau) e^{im\omega t} Y_s^{(k)}(t, \varepsilon).$$

It can then be shown that the operators $V^{(k)}(t, \varepsilon)$ give asymptotic approximations to the operator $U(t, \varepsilon)$ on the interval $0 \leq \tau = \varepsilon t \leq l$.

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Note: Figure translations are in progress. See original paper for figures.

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