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**Abstract**

**Full Text**

**A. Kh. Gelig**

**ON THE STABILITY OF MOTION OF SYSTEMS WITH A NONUNIQUE EQUILIBRIUM POSITION**

*(Presented by Academician V. I. Smirnov on June 7, 1962)*

The proposed article gives sufficient conditions for the stability, as a whole, of stationary sets of nonlinear discontinuous systems which, generally speaking, possess a nonunique equilibrium position. The results are based on a method developed by V. M. Popov <sup>(1)</sup> for the investigation of nonlinear continuous systems with a unique equilibrium position.

Consider the system

$$\dot{x} = Ax + a\varphi(\sigma); \tag{1}$$

$$\sigma = b^*x, \tag{2}$$

where  $A$  is a constant square matrix, all of whose eigenvalues have negative real parts;  $a$  and  $b$  are constant vectors;  $x$  is a vector-function of the time  $t$ ;  $\varphi(\sigma)$  is a scalar function;  $*$  denotes transposition.

It is assumed that  $\varphi(\sigma)$  is continuous everywhere except at isolated points  $\sigma_k$ , at which it has discontinuities of the first kind, and

$$\varphi(\sigma)\sigma > 0 \quad \text{for } \sigma \neq 0; \tag{3}$$

$$\varphi(\sigma_k - 0)\varphi(\sigma_k + 0) > 0 \quad \text{for } \sigma_k \neq 0; \tag{4}$$

$$|\varphi(\sigma)| < \text{const} \quad \text{for } -\infty < \sigma < +\infty. \tag{5}$$

Following the point of view adopted in <sup>(2)</sup>, we shall say that the vector-function  $x(t)$  is a solution of the Cauchy problem

$$x|_{t=0} = x_0 \tag{6}$$

for the system (1), (2) on the interval  $[0, T]$ , if it is absolutely continuous, satisfies (6), and for almost all  $t \in [0, T]$ , for any  $\delta > 0$ , the vector  $\dot{x}$  belongs to the

smallest convex closed set (of  $n$ -dimensional space) containing all values of the vector-function  $\{Ax + a\varphi(\sigma)\}_{x=x'}$ , when  $x'$  ranges over almost the entire  $\delta$ -neighborhood of the point  $x(t)$  in the space  $\{x\}$ .

Under the assumptions made, according to <sup>(2)</sup>, such a solution of the system (1), (2) exists on the entire ray  $[0, +\infty)$ .

The stationary points of the system (1), (2) are determined by the formula

$$x = -A^{-1}a\varphi(\sigma), \quad (7)$$

where  $\sigma$  are the roots of the equation

$$\sigma + (b, A^{-1}a)\varphi(\sigma) = 0. \quad (8)$$

Consider the function

$$\chi(p) = b^*(A - pE)^{-1}a. \quad (9)$$

If  $\chi(0) < 0$ , then, as is easy to verify, even for the linear function  $\varphi(\sigma) = \chi\sigma$ , with sufficiently large  $\chi > 0$ , system (1), (2) will be unstable.

For  $\chi(0) \geq 0$  the following three cases may occur:

- 1)  $\varphi(\sigma)$  is continuous at zero;
- 2)  $\varphi(\sigma)$  is discontinuous at zero and  $\chi(0) > 0$ ;
- 3)  $\varphi(\sigma)$  is discontinuous at zero and  $\chi(0) = 0$ .

Analyzing equations (7), (8), we arrive at the conclusion that in the first two cases system (1), (2) has the unique stationary point  $x = 0$ , while in the third case the system possesses an entire "interval of rest"

$$x = -A^{-1}a\xi, \quad \xi \in [\varphi(-0), \varphi(+0)]. \quad (10)$$

In the following theorems sufficient conditions are given for stability in the large of these stationary sets.

**Theorem 1.** *If: 1) there exists an  $\alpha \geq 0$  such that for  $-\infty < \omega < +\infty$*

$$\operatorname{Re}(1 + i\alpha\omega)\chi(i\omega) \geq 0; \quad (11)$$

- 2)  $\varphi(\sigma)$  satisfies conditions (3)–(5) and is continuous at zero, then the equilibrium position  $x = 0$  of system (1), (2) is stable in the large.

In the proof of Theorem 1, <sup>(3)</sup> and the method developed by V. M. Popov <sup>(1)</sup> are used. Under a strict inequality in (11) and continuous  $\varphi(\sigma)$ , this result follows from <sup>(1)</sup>. However, the possibility of equality in (11) is very substantial for the class of systems with nonunique equilibrium position.

Theorem 1 is easily generalized to the case when system (1) contains several nonlinear functions.

**Theorem 2.** *If: 1) the first condition of Theorem 1 is fulfilled and  $\chi(i\omega) \neq 0$  for all real  $\omega \neq 0$ ; 2)  $\varphi(\sigma)$  satisfies conditions (3)–(5) and has a discontinuity of the first kind at zero; 3)  $\chi(0) > 0$ , then the equilibrium position  $x = 0$  of system (1), (2) is stable in the large.*

**Theorem 3.** *If: 1) the first condition of Theorem 2 is fulfilled; 2) the function  $\varphi$  satisfies condition 2) of Theorem 2 and, moreover, there exists an  $\varepsilon_0 > 0$  such that*

$$\begin{aligned} \varphi(\sigma) &> \varphi(+0) & \text{for } 0 < \sigma \leq \varepsilon_0, \\ \varphi(\sigma) &< \varphi(-0) & \text{for } -\varepsilon_0 \leq \sigma < 0; \end{aligned} \quad (12)$$

3)  $\chi(0) = 0$ , then the stationary set (10) of system (1), (2) is stable in the large.

The proof of the last two theorems is based on the following simple lemma.

**Lemma.** *If  $\lim_{t \rightarrow +\infty} \alpha(t) = 0$ ,  $\alpha'(t)$  is absolutely continuous and  $\sup_{t > 0} |\alpha''(t)| < \text{const}$ , then  $\lim_{t \rightarrow +\infty} \alpha'(t) = 0$ .*

Condition (12) has a clear physical interpretation. If the function  $\varphi(\sigma)$  expresses the dependence of the friction force on velocity, then (12) means that the friction is not of a “breakaway” character.

It is also of interest to consider systems in which this condition is violated. Sufficient conditions for stability in the large of the stationary sets of such systems are given by the following corollary.

**Corollary.** *If there exists a  $\mu > 0$  such that: 1) the eigenvalues of the matrix  $A - \mu ab^*$  have negative real parts; 2) the function*

$$\chi_\mu(p) = \frac{\chi(p)}{1 - \mu\chi(p)}$$

has no purely imaginary roots and satisfies inequality (11) for  $\omega \in (-\infty, +\infty)$  and some  $\alpha \geq 0$ ; 3) the function  $\varphi(\sigma)$  satisfies the second condition of Theorem 2 and, moreover, there exists an  $\varepsilon_0 < 0$  such that

$$\varphi(\sigma) > \varphi(+0) - \mu\sigma \quad \text{for } 0 < \sigma \leq \varepsilon_0,$$

$$\varphi(\sigma) < \varphi(-0) - \mu\sigma \quad \text{for } -\varepsilon_0 \leq \sigma < 0,$$

- 4)  $\chi(0) = 0$ , then the stationary set (10) of the system (1), (2) is stable as a whole.

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## REFERENCES

<sup>1</sup> V. M. Popov, *Studii și cercetări energ. Acad. RPR*, **10**, 1, 159 (1960). <sup>2</sup> A. F. Filippov, *Matem. sborn.*, **51** (93), 1, 99 (1960). <sup>3</sup> E. Kamke, *The Lebesgue-Stieltjes Integral*, Moscow, 1959.

*Note: Figure translations are in progress. See original paper for figures.*

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