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Abstract

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ON ONE-SIDED BOUNDARY-VALUE PROBLEMS IN THE THEORY OF ANALYTIC FUNCTIONS

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A characteristic feature of the Riemann boundary-value problem and of its various generalizations ([1,2]) is that on the boundary of the domain the limiting values of the sought analytic functions, taken respectively on the left and right sides of the boundary, are linearly related. A substantially new class of boundary-value problems for analytic functions is formed by the so-called one-sided problems, in which the unknown functions are sought from a linear relation between their limiting values prescribed on one side of the boundary. F. D. Gakhov and Yu. I. Cherskii ([3,4]) encountered problems of this kind (and partly investigated them) in connection with the study of special integral equations of convolution type, and I. Kh. Khairullin ([5]) did so in considering one class of infinite systems of linear algebraic equations.

Let us take a Lyapunov contour L bounding a simply connected domain D^+. By D^- we denote the complement of $D^+ + L$ in the complete plane. Suppose that on L a function $\alpha_+(t)$ is given which maps L one-to-one onto itself with preservation of the orientation on L, $\alpha'_+(t) \neq 0$ on L. By the letter t we denote the complex coordinate of a variable point of L. Let $\alpha_-(t)$ be a function with analogous properties, but changing the orientation on L. In the present article some results are announced which were obtained by the authors in studying the following one-sided boundary-value problems.

Find functions $\varphi(z)$ and $\psi(z)$, analytic in D^{+*}, whose angular limiting values are square integrable on L, representable by the Cauchy formula, and satisfying almost everywhere on L one of the boundary conditions

$$\varphi[\alpha_+(t)] = G(t)\psi(t) + g(t); \tag{1}$$

$$\varphi[\alpha_-(t)] = G(t)\overline{\psi(t)} + g(t). \tag{2}$$

For $\varphi(z) \equiv \psi(z)$ we obtain the problems

$$\varphi[\alpha_+(t)] = G(t)\varphi(t) + g(t) \quad \text{on } L; \tag{3}$$

$$\varphi[\alpha_-(t)] = G(t)\overline{\varphi(t)} + g(t) \quad \text{on } L. \tag{4}$$

Find a piecewise analytic function $\{\varphi^+(z), \varphi^-(z)\}$ from one of the boundary conditions

$$\varphi^+[\alpha_-(t)] = G(t)\varphi^-(t) + g(t); \tag{5}$$

$$\varphi^+[\alpha_+(t)] = G(t)\overline{\varphi^-(t)} + g(t). \tag{6}$$

The functions $G(t)$ and $\alpha'_\pm(t)$ are assumed to satisfy a Hölder condition on L, and $g(t) \in L_2$.

* Problems (1)–(4) may also be posed for the domain D^-.

§ 1. Using the conditions that $\varphi(t) = G[\beta(t)]\psi[\beta(t)]+g[\beta(t)]$, where $\alpha_+[\beta(t)] \equiv t$, and that $\psi(t)$ are the boundary values of functions analytic in the domain D^+, from the boundary-value problem (1) we arrive at a Fredholm integral equation of the first kind with a quasi-regular kernel (([1], p. 129))

$$\frac{1}{2\pi i}\int_L \left[\frac{G(t)}{\tau - t} - \frac{G(\tau)\alpha'_+(\tau)}{\alpha_+(\tau) - \alpha_+(t)}\right]\psi(\tau)\,d\tau = \{g[\beta(\xi)]\}_{\xi=\alpha_+(t)}. \tag{7}$$

By $h^\pm(t)$ we mean the operators

$$h^\pm(t) = \pm\frac{1}{2}h(t) + \frac{1}{2\pi i}\int_L \frac{h(\tau)}{\tau - t}\,d\tau.$$

Further in this paragraph we shall assume that the functions $G(t)$, $\alpha'_\pm(t)$, as well as the angle formed by the tangent to L with any direction, satisfy a Hölder condition with exponent $\mu > 1/2$. Then the kernel of equation (7) belongs on L to the class L_2, and for equation (7) the following assertion is valid, which is an extension of the well-known theorem of Picard ([6]) to equations with a complex nonsymmetric kernel. For the solvability of the integral equation

$$\int_L K(t,\tau)\varphi(\tau)\,d\tau = f(t), \quad t \in L, \tag{7'}$$

it is necessary and sufficient that, in the mean,

$$\sum_k a_k\lambda_k\mu_k(s) = \int_0^S \overline{f[t(s_1)]}\,K(s,s_1)\,ds, \tag{8}$$

where

$$a_k = \int_0^S f(t(s))\overline{\mu_k(s)}\,ds = \frac{1}{\lambda_k}\int_0^S f(t(s))\overline{\nu_k(s)}\,ds;$$

λ_k are the eigenvalues of the kernel $K[t(s),\tau(\sigma)]$; $\mu_k(s)$ and $\nu_k(s)$ are orthonormal systems of eigenfunctions of the right and left iterated kernels, respectively,

$$K_R(s,\sigma) = \int_0^S \widetilde{K}(s,\xi)\overline{\widetilde{K}(\sigma,\xi)}\,d\xi$$

and

$$K_L(s,\sigma) = \int_0^S \overline{\widetilde{K}(\xi,s)}\widetilde{K}(\xi,\sigma)\,d\xi, \qquad \widetilde{K}(s,\xi) \equiv K[t(s),\tau(\xi)].$$

The general solution of equation $(7')$ is given by the formula

$$\varphi[t(s)] = \frac{1}{\tau'(s)}\left[\sum_k a_k\lambda_k\mu_k(s) + \sum_j c_j\tilde{\mu}_j(s)\right].$$

Here c_j are arbitrary constants, and the system $\{\tilde{\mu}_j(s)\}$ completes the system of eigenfunctions $\{\mu_k(s)\}$ to a closed one. In the case of a closed kernel $K(t,\tau)$, all $c_j = 0$, and equation $(7')$ has a unique solution in L_2.

If equation (7) is unsolvable, then problem (1) likewise has no solutions. Suppose that equation (7) is solvable, and let $\lambda(t)$ be its general solution. Then on L the identity

$$\{G[\beta(\xi)]\lambda[\beta(\xi)] + g[\beta(\xi)]\}^+_{\xi=\alpha_+(t)} \equiv G(t)\lambda^+(t) + g(t)$$

holds.

Theorem 1. *The one-sided boundary-value problem (1) is solvable if condition (8) is fulfilled, and the general solution of the problem has the form*

$$\psi(z) = \lambda^+(z), \qquad \varphi(z) = \frac{1}{2\pi i}\int_L \frac{G[\beta(\tau)]\lambda[\beta(\tau)] + g[\beta(\tau)]}{\tau - z}\,d\tau, \tag{9}$$

where $\lambda(t)$ is the general solution of the integral equation (7).

Theorem 2. *The homogeneous problem (1) has no nontrivial solutions in D^+ if every solution $\lambda(t)$ of the homogeneous equation (7) is the boundary value of a function analytic in D^- and vanishing at infinity, i.e. $\lambda^+(t) \equiv 0$ on L. If $\lambda^+(t) \not\equiv 0$, then the homogeneous problem (1) is nontrivially solvable and its* *general solution is given by formulas (9), where $g(t) \equiv 0$, and $\lambda(t)$ is the general solution of the homogeneous equation (7).*

We note that if the general solution of the inhomogeneous equation (7) satisfies the condition $\lambda^+(t) \equiv 0$, then problem (1) is solvable, provided that $g^-[\beta(t)] = 0$ on L.

Results analogous to Theorems 1 and 2 are also valid for problems (2), (5), (6). For example, the following is valid.

Theorem 3. The boundary-value problem (6) is solvable if condition (8) is satisfied. The general solution of the problem has the form

$$\varphi^-(z) = -\lambda^-(z), \qquad \varphi^+(z) = \frac{1}{2\pi i}\int_L \frac{G[\beta(\tau)]\,\overline{\lambda[\beta(\tau)]} + g[\beta(\tau)]}{\tau - z}\,d\tau,$$

where $\lambda(t)$ is the general solution of the Fredholm integral equation of the first kind

$$\frac{1}{2\pi i}\int_L \left[\frac{\overline{G(t)}}{\tau - t} - \frac{\overline{G(\tau)}\,\alpha'(\tau)\tau^2(\sigma)}{\alpha(\tau) - \alpha(t)}\right]\varphi^-(\tau)\,d\tau = -\overline{\{g[\beta(\xi)]\}^-}_{\xi=\alpha_+(t)}.$$

Let us consider boundary-value problem (3) under the condition that $\alpha_-[\alpha_+(t)] = t$ on L.

Theorem 4. If $G(t)G[\alpha_+(t)] \neq 1$, then the homogeneous problem (3) has only the trivial solution. If $G(t)G[\alpha_+(t)] = 1$, then linearly independent nontrivial solutions of problem (3) are given by the formulas $\varphi_j(z) = \lambda_j^+(z)$, where $\lambda_j(t)$ are linearly independent solutions of the homogeneous equation (7) satisfying the condition

$$\{\lambda(t) - G[\alpha_+(t)]\lambda[\alpha_+(t)]\}^+ \equiv 0. \tag{10}$$

Theorem 5. Let $G[\alpha_+(t)]g(t) + g[\alpha_+(t)] \neq 0$ on L. Then the one-sided problem (3) is solvable and has a unique solution if, in addition, $G(t)G[\alpha_+(t)] \neq 1$ on L and the function

$$k(t) = \frac{G[\alpha_+(t)]g(t) + g[\alpha_+(t)]}{1 - G(t)G[\alpha_+(t)]}$$

is the boundary value of a function analytic in D^+. If $G[\alpha_+(t)]g(t) + g[\alpha_+(t)] \neq 0$, but $G(t)G[\alpha_+(t)] = 1$, or $k^-(t) \neq 0$, then problem (3) is not solvable. If, however, $G[\alpha_+(t)]g(t) + g[\alpha_+(t)] = 0$, then a solution of problem (3), $\varphi(z) = \lambda^+(z)$, exists when conditions (8), (10) and the condition $\{g[\alpha_+(t)]\}^+ = 0$ on L are satisfied.

Analogous assertions hold for problem (4).

§ 2. We give several simple results clarifying the influence of the properties of $G(t)$ and $\alpha_\pm(t)$ on the solvability of one-sided problems and on the character of their general solutions. Consider the case where, in problems (1), (3), (5), the functions $\alpha(t)$, and in problems (2), (4), (6) the functions $\overline{\alpha(t)}$, are analytically continuable into D^+ with a finite number of poles. Then, from the properties of the transformation $\alpha(t)$ indicated above, it follows that the function $\alpha(z)$ ($\overline{\alpha(z)}$) is single-valued in D^+ and maps D^+ one-to-one and conformally onto the domain bounded by the curve $\xi = \alpha(t)$ ($\xi = \overline{\alpha(t)}$), $t \in L$. Consequently, boundary-value problem (1) in the case under consideration is equivalent to a problem of the form:

Find two functions $\varphi_1(z)$, $\psi_1(z)$ analytic in D^+ from the boundary condition

$$\varphi_1(t) = G(t)\psi_1(t) + g(t) \quad \text{on } L. \tag{1'}$$

For problem ($1'$) the following obvious assertions are valid:

1) If $G(t)$ is not the boundary value of a function meromorphic in D^+, then the homogeneous problem ($1'$) has only the trivial solution, while the nonhomogeneous problem ($1'$) can have at most one solution.

2) If $G(t)$ has the form $G(t) = g(t)/h(t)$, where $g(z)$ and $h(z)$ are functions analytic in D^+, representable by the Cauchy formula, and whose boundary values belong to L_2 on L, then, when condition (8) is satisfied, the general solution of problem ($1'$) is given by the formulas

$$\psi_1(z) = \Omega(z)h(z) + \psi_0(z), \qquad \varphi_1(z) = \Omega(z)g(z) + \varphi_0(z),$$

where $\varphi_0(z)$, $\psi_0(t)$ is any particular solution of problem ($1'$), and $\Omega(z)$ is an arbitrary analytic function in D^+ with boundary values from L_2.

Analogous results are obtained for the boundary-value problems (2), (5), and (6).

Let us note the case when, in the boundary condition (6), $\alpha_+(t) \equiv t$. If t is a boundary value of a function analytic in D^-, then problem (6) has the same character as problem ($1'$). If, moreover, $G(t) \equiv 1$, $g(t) \equiv 0$, and L is a circle or a lemniscate, then we obtain a problem that was considered by another method by A. I. Markushevich [7]. The result for the case when D^+ is a disk, $\alpha_+(t) \equiv t$, and $G(t)$ and $g(t)$ are boundary values of functions analytic in D^+, was formulated in a paper of L. G. Mikhailov [8].

Let us consider the boundary-value problems (3) and (4). Without loss of generality, we shall assume that D^+ is the disk $|z| < 1$. Then, introducing as before the condition of analytic continuability of $\alpha_+(t)$, we obtain that $\alpha_+(t) = e^{i\theta}(t-a)/(1-\bar{a}t)$, $|a| < 1$. If the homogeneous problem corresponding to the solvable problem (3) has nontrivial solutions, then the general solution of problem (3) has the form

$$\varphi(z) = \varphi_0(z) + \varphi_1(z)\Omega(z),$$

where $\varphi_0(z)$ is a particular solution of the nonhomogeneous problem (3); $\varphi_1(z)$ is any nontrivial solution of the homogeneous problem (3); $\Omega(z)$ is an arbitrary meromorphic function in $|z| < 1$ satisfying the conditions: a) the boundary values $\Omega^+(t) \in L_2$ on $|t| = 1$; b) $\Omega(z)$ is an automorphic function with respect to the group $\{t, \alpha(t), \alpha[\alpha(t)], ...\}$; c) $\Omega(z)\varphi_1(z)$ is an analytic function in $|z| < 1$.

Let us consider problem (4), assuming that $\alpha\_(t)$ is a real function on $|t| = 1$ (i.e., $\alpha(t)$ is real for $t = \pm 1$), satisfying the condition $\alpha\_[\alpha\_(t)] = t$. Assuming the analytic continuability of $\alpha\_(t)$ inside $|z| < 1$, we obtain that $\alpha\_(t) = (a - t)/(1 - \bar{a}t)$, $\operatorname{Im} a = 0$, $|a| > 1$. Introduce the functions

$$u(z) = \frac{1}{2}[\Psi(z) + \overline{\Psi}(z)], \qquad v(z) = \frac{1}{2i}[\Psi(z) - \overline{\Psi}(z)],$$

where $\Psi(z)$ is a meromorphic function in $|z| < 1$. The functions $u(z)$, $v(z)$ are real meromorphic functions.

Let problem (4) be solvable, and let $\varphi_0(z)$ be its particular solution. Let $\varphi_1(z)$ be some nontrivial solution of the homogeneous problem (4). Then the general solution of problem (4) has the form

$$\varphi(z) = \varphi_0(z) + \varphi_1(z) \left\{ u(z) + i \left[f_0(z) - f_0\left(\frac{1 - az}{a - z}\right)\right] w(z) \right\}, \qquad (11)$$

where

$$\left[f_0(z) - f_0\left(\frac{1 - az}{a - z}\right)\right] w(z) = v(z);$$

$u(z)$ and $w(z)$ are real meromorphic functions automorphic with respect to the group $\{t, (1 - at)/(a - t)\}$ and such that the second term in (11) is an analytic function in $|z| < 1$; $f_0(z)$ is a fixed real meromorphic function, nonautomorphic with respect to $\{t, (1 - at)/(a - t)\}$; $u(t)$, $w(t)$, $f_0(t) \in L_2$ on $|t| = 1$.

The question of the character of the general solution of the boundary-value problems (1)–(6) for the case of functions $\alpha(t)$ (or $\overline{\alpha(t)}$) that are not analytically continuable inside the domain D^+ remains open.

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Note: Figure translations are in progress. See original paper for figures.