SOLUTION OF CERTAIN MATRIX INEQUALITIES OCCURRING IN THE THEORY OF AUTOMATIC CONTROL

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Abstract

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SOLUTION OF CERTAIN MATRIX INEQUALITIES OCCURRING IN THE THEORY OF AUTOMATIC CONTROL

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1°. We shall denote square matrices by capital Latin letters, column vectors by lowercase Latin letters, and numbers by Greek letters. An asterisk denotes Hermitian conjugation, so that ab^* is a matrix, $b^*a = (a, b)$ is the scalar product. The notation $H > 0$ means that H is a Hermitian positive-definite matrix. I is the identity matrix.

Consider the following problems:

(I_ν). Given A, a, b; the eigenvalues of A lie in the left half-plane. For $H = H^*$ define

$$G = -(A^*H + HA), \qquad g = -(Ha + b). \tag{1}$$

It is required to specify conditions under which the quadratic inequality with respect to the matrix $H = H^*$

$$G - gg^* > 0 \tag{2}$$

has a solution.

(II_ν). Given $B, c \neq 0, d \neq 0$; the eigenvalues of B lie in the left half-plane. It is required to specify conditions under which there exists $X = X^*$, satisfying the relations

$$-Y \equiv B^*X + XB < 0, \qquad Xc + d = 0. \tag{3}$$

The index ν in both problems denotes the order of the corresponding vectors and matrices. By simple transformations one can reduce problem (I_ν) to ($\mathrm{II}_{\nu+1}$) and conversely (see below, paragraphs $7^\circ, 8^\circ$).

In applications the data of the problems (I_ν), (II_ν) depend on parameters, which must be chosen so that the corresponding problem has a solution. In this connection, rational solutions of the problems (I_ν), (II_ν) are desirable, i.e. such solutions that can be reduced to checking a finite number of conditions $\xi_\mu > 0,\ \eta_\mu = 0$, where ξ_μ, η_μ are polynomials with respect to the data of the problems (i.e. with respect to the real and imaginary parts of the elements A, a, b, B, c, d).

For the sake of generality and convenience of the solution, we assume the elements of the vectors and matrices to be complex, although in applications they are real.

2°. The problems $(\mathrm{I}_\nu), (\mathrm{II}_\nu)$ lead to problems of finding, in a certain sense, optimal conditions for stability in the large of nonlinear differential equations with one nonlinearity of class (A) ([1−5]), to problems with a fixed nonlinearity of the type considered in ([6]), and some others. The Lurie method ([1,2]) gives rational sufficient conditions for the solvability of problem (I_ν). For $\nu > 2$ these conditions are also necessary ([5]); it can be shown that for $\nu > 2$ these conditions do not coincide with the necessary ones*. Let us note the effective sufficient conditions of Lefschetz ([7]), which may be regarded as sufficient conditions for the solvability of problem (I_ν). In the work ([8]) V. M. Popov derived a partial condition for stability in the large of systems with a nonlinearity of class (A), encompassing all conditions that can be

* Thus, the problem formulated in ([5]), p. 129, has a negative solution.

obtained by means of a Lyapunov function of the form "quadratic form plus an integral of the nonlinearity." Theorem 1 (see below), with a small addition, gives a new proof of this Popov condition together with an answer to the inverse problem posed in ([8]), p. 972. Namely, when Popov' s frequency condition ([8]) is satisfied, the system has a Lyapunov function of the indicated form.

3°. Introduce the following notation:

$$\begin{gathered} A_\omega = A - i\omega I, \qquad a_\omega = A_\omega^{-1} a, \qquad b_\omega = A_\omega^{*-1} b, \\ \varphi_I(\omega) = 1 + 2\,\mathrm{Re}(a_\omega, b), \qquad B_\omega = B - i\omega I, \qquad c_\omega = B_\omega^{-1} c, \\ \varphi_{\mathrm{II}}(\omega) = \mathrm{Re}(c_\omega, d), \qquad \chi = -(c, d), \qquad \beta = \mathrm{Re}(Bc, d). \end{gathered} \tag{4}$$

We shall call the functions $\varphi_I(\omega)$, $\varphi_{\mathrm{II}}(\omega)$ the characteristics of problems (I_ν), (II_ν), respectively.

Theorem 1. In order that inequality (2) have a solution $H = H^*$, it is necessary and sufficient that $\varphi_I(\omega) > 0$ for $-\infty < \omega < +\infty$.

Theorem 2. In order that there exist a matrix $X = X^*$ satisfying relations (3), it is necessary and sufficient that: 1) χ be real; 2) $\beta \neq 0$; 3) $\varphi_{\mathrm{II}}(\omega) > 0$ for $-\infty < \omega < +\infty$.

Theorem 3. In order that there exist a matrix $X = X^*$ satisfying relations (3) and the inequality $Y > \varepsilon I$ with a given $\varepsilon > 0$, it is necessary that: 1) χ be real;

2) $\varepsilon_0 = 2\,\mathrm{Inf}\,\varphi_{\mathrm{II}}(\omega)/|c_\omega|^2 > 0$; 3) $\varepsilon \le \varepsilon_0$, and it is sufficient that conditions 1), 2) and 3′ $\varepsilon < \varepsilon_0$ be fulfilled.

Obviously, Theorems 1, 2 and 3 give rational solutions*.

4°. **Necessity of the conditions of Theorems 1-3.** We have

$$A_\omega^* H + H A_\omega = -G, \qquad (Ga_\omega, a_\omega) = -2\,\mathrm{Re}(Ha, a_\omega) = 2\,\mathrm{Re}(b, a_\omega) + 2\,\mathrm{Re}\,\zeta,$$

where $\zeta = (g, a_\omega)$. From (2) and the first relation (1) we successively derive

$$(Ga_\omega, a_\omega) > |\zeta|^2, \qquad \varphi_I(\omega) > 1 + \zeta^{2**}.$$

The necessity of the conditions of Theorems 2 and 3 follows from the relations $\chi = (Xc, c)$, $\beta = (Yc, c)$, $2\varphi_{\mathrm{II}}(\omega) = (Yc_\omega, c_\omega) > \varepsilon |c_\omega|^2$.

5°. **Sufficiency of the conditions of Theorem 3** follows from the sufficiency in Theorem 2 (for matrices of fixed order ν). Setting $X = X_0 + \varepsilon H_0$, where $B^* H_0 + H_0 B = -I$, we obtain that problem (II_ν) has the required solution if there exists $X_0 = X_0^*$ satisfying the conditions

$$B^* X_0 + X_0 B = -Y_0 < 0, \qquad X_0 c + d_0 = 0,$$

where $d_0 = \varepsilon H_0 c + d$. From conditions 1), 2), 3′ of Theorem 3 it follows that conditions 1), 2), 3) of Theorem 2 are fulfilled and, consequently, that X_0 exists.

6°. We shall write $\{A\} = \mu$ if the matrix A has μ eigenvalues in the left half-plane and has no purely imaginary or zero eigenvalues.

Lemma 1. Consider a matrix of order $\nu + 1$

$$\widetilde{B} = \begin{pmatrix} A & p \\ q^* & \alpha \end{pmatrix}. \tag{5}$$

Let $\mathrm{Re}\,\alpha \neq 0$ and let e be the vector with $\nu + 1$ components $0, \dots, 0, 1$. Put

$$\chi_I(\omega) = ((A - i\omega)^{-1} p, q) - \alpha, \qquad \chi_{\mathrm{II}}(\omega) = -((\widetilde{B} - i\omega I)^{-1} e, e).$$

Then the conditions: a_1) $\mathrm{Re}\,\chi_I(\omega) > 0$ for $-\infty < \omega < +\infty$, b_1) $\{A\} = \mu$ are equivalent to the conditions: a_2) $\mathrm{Re}\,\chi_{\mathrm{II}}(\omega) > 0$ for $-\infty < \omega < +\infty$, b_2) $\{\widetilde{B}\} = \mu + 1$.

Proof. Solving the equation

$$(\widetilde{B} - i\omega I)\begin{pmatrix} x \\ \xi \end{pmatrix} = e,$$

we find

$$\xi = \chi_{\text{II}} = (\chi_I + i\omega)^{-1}.$$

Therefore a_2) follows from a_1). From a_1), b_1) and the relation

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* Another, less convenient, rational solution of problems (I_ν), (II_ν) was reported by the author at V. V. Nemytskii' s seminar at Moscow State University in the spring of 1961.

** Note that the necessity of the condition $\varphi_I(\omega) > 0$ is proved essentially in ([3],[5]). The necessary condition (3.2) ([5]) for complex A, a, b, in the notation adopted here, has the form $\Gamma^2 = 1 + 2\,\text{Re}(b, A^{-1}a) > 0$. Since problem ($\text{I}_\nu$) is not changed when A is replaced by A_ω, we have hence $\varphi_I(\omega) > 0$.

$$\chi_{11} = -\det(A - i\omega I)/\det(\widetilde{B} - i\omega I)$$

we conclude that $\widetilde{B}$ has no purely imaginary eigenvalues. For $|\omega| \to \infty$ we have

$$\text{Re}\,\chi_{11} = -\text{Re}\,a/\omega^2 - [1 + O(\omega^{-1})],$$

i.e. $\text{Re}\,a < 0$, and

$$\Delta \operatorname{Arg} \chi_{11}(\omega)\big|_{-\infty}^{+\infty} = -\pi.$$

Therefore

$$\Delta \operatorname{Arg} \det(\widetilde{B} - i\omega I)\big|_{-\infty}^{+\infty} = \pi[(\nu+1) - (\mu+1)],$$

i.e. $\{\widetilde{B}\} = \mu + 1$. The converse assertion is proved analogously.

7°. **The sufficiency in Theorem 1** follows from the sufficiency in Theorem 2. Consider problem (II_ν) for

$$B = \begin{pmatrix} A & a \\ b^* & -1/2 \end{pmatrix}, \qquad c = e, \qquad d = -e.$$

Applying Lemma 1, we find that the conditions of Theorem 2 are satisfied. Therefore there exists a solution of problem ($\text{II}_{\nu+1}$)

$$X = \begin{pmatrix} H & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easy to verify that the matrix H is a solution of problem (I_ν).

8°. **Proof of sufficiency in Theorem 2.** For $\nu = 1$ the assertion of Theorem 2 is verified directly. Suppose that sufficiency has been proved for matrices of order ν. Consider problem $(\mathrm{II}_{\nu+1})$ under the assumption that the conditions of Theorem 2 are satisfied. From the formula

$$\pi\chi = \lim_{\Omega\to+\infty}\int_{-\Omega}^{\Omega}\varphi_{11}(\omega)\,d\omega$$

it follows that $\chi > 0$. Therefore the matrix S, whose last column is equal to c, and whose first ν columns are any basis in the subspace orthogonal to d, is nonsingular. Multiplying the first relation (3) on the left by S^*, on the right by S, and the second on the left by S^*, we pass to the equivalent problem

$$-\widetilde{Y} \equiv \widetilde{B}^*\widetilde{X} + \widetilde{X}\widetilde{B} < 0, \qquad \widetilde{X}e = \chi e, \tag{6}$$

where $\widetilde{B} = S^{-1}BS$. Since

$$\varphi_{11}(\omega) = \frac{\beta}{\omega^2}[1 + O(\omega^{-1})] \quad \text{as } |\omega| \to \infty,$$

we have $\beta > 0$. Representing $\widetilde{B}$ in the form (5), we find that

$$\operatorname{Re} a = -\beta/\chi < 0, \qquad \operatorname{Re}\chi_{11}(\omega) = \varphi_{11}(\omega)/\chi > 0.$$

By Lemma 1 we have $\operatorname{Re}\chi_1(\omega) > 0$, $\{A\} = \nu$. The second relation (6) means that $\widetilde{X}$ has the form

$$\widetilde{X} = \begin{pmatrix} H & 0 \\ 0 & \chi \end{pmatrix}.$$

From the first relation (6) it follows that the required matrix H is a solution of problem (I_ν), where

$$a = p/\sqrt{\beta}, \qquad b = \chi q/\sqrt{\beta}.$$

The characteristic of this problem is

$$\varphi_I(\omega) = \frac{\chi}{\beta}\operatorname{Re}\chi_1(\omega) > 0.$$

Consider the first case, when

$$\operatorname{Re}(A_\omega^{-1}a, b) < 0$$

for some ω. Then there exists τ such that

$$\operatorname{Inf}\varphi_I(\omega) = \varphi_I(\tau).$$

Consider the auxiliary problem

$$-Y_1 = A^*X_1 + X_1A < 0, \qquad X_1a_\tau + b_\tau - \delta H_0 a_\tau = 0, \tag{7}$$

where H_0 is the solution of the equation

$$A^*H_0 + H_0A = -I,$$

$\delta > 0$, and a_τ, b_τ are determined from (4). By the induction hypothesis and item 4° we may apply Theorem 3. The value

$$\chi_{\text{aux}} = (A_\tau^{-2}a, b) + \delta(H_0 a_\tau, a_\tau)$$

is real, since $\varphi_I'(\tau) = 0$. The characteristic of the auxiliary problem is transformed to the form

$$\varphi_{\text{aux}} = \frac{[\varphi_I(\omega) - \varphi_I(\tau)]}{(\tau-\omega)^2} + \frac{\delta}{2}|A_\omega^{-1}a_\tau|^2.$$

Therefore $\varepsilon_0 \geq \delta/2$, and by Theorem 3 there exists a matrix $X_1 = H$ such that

$$Y_1 > \frac{\delta}{3}I.$$

We shall show that the matrix H is a solution of problem (I_ν) provided that the number $\delta > 0$ is chosen sufficiently small. After some calculations we obtain, using the second relation (7),

$$(G^{-1}g, g) = -2\operatorname{Re}(a_\tau, b) + \delta^2(G^{-1}h, h) < 1 - \varphi_I(\tau) + 3\delta|h|^2,$$

where $h = A_\tau^* H_0 a_\tau$. For

$$0 < \delta < \varphi_I(\tau)/3|h|^2$$

we obtain

$$(G^{-1}g, g) < 1,$$

which, under the condition $G = -Y_1 > 0$, is equivalent to (2). For the first case the proof is complete.

In the second case, when $\operatorname{Re}(a_\omega, b) > 0$, the auxiliary problem has the form $-Y_1 \equiv A^*X_1 + X_1A < 0$, $X_1a + b - \delta H_0 a = 0$, where $0 < \delta < |H_0a|^2/3$. All the arguments are only simplified. The number $\varkappa_{\text{aux}} = -(a, b) - \delta(H_0a, a)$ is real, since as $|\omega| \to \infty$, from the condition $\operatorname{Re}(a_\omega, b) = -\operatorname{Im}(a, b)/\omega + O(\omega^{-2}) > 0$ it follows that $\operatorname{Im}(a, b) = 0$. The characteristic of the auxiliary problem has the form $\varphi_{\text{aux}} = \operatorname{Re}(a_\omega, b) + \dfrac{\delta}{2}|a_\omega|^2$, i.e. $\varepsilon_0 \geqslant \delta/2$. By Theorem 3 there exists $X_1 = X_1^*$ such that $Y_1 > \dfrac{\delta}{3}I$. For $H = X_1$ we have $G = -Y_1 > 0$, $-g = \delta H_0 a$, $(G^{-1}g, g) < 1$. Theorem 2 is proved.

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REFERENCES CITED

[1] A. I. Lur' e, *Some Nonlinear Problems in the Theory of Automatic Control*, Moscow—Leningrad, 1951. [2] A. M. Letov, *Stability of Nonlinear Controlled Systems*, Moscow—Leningrad, 1955. [3] V. A. Yakubovich, DAN, 117, No. 1

(1957). [4] V. A. Yakubovich, DAN, 121, No. 6 (1958). [5] V. A. Yakubovich, Vestn. LGU, No. 7, issue 2 (1960). [6] V. A. Yakubovich, DAN, 135, No. 1 (1960). [7] S. Lefschetz, RIAS technical report 60–9, 1960. [8] V. M. Popov, Avtomatika i telemekhanika, 22, No. 8 (1961).

Note: Figure translations are in progress. See original paper for figures.