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Abstract

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ON DYNAMICAL SYSTEMS WITHOUT UNIQUENESS AS SEMIGROUPS OF MULTIVALUED MAPPINGS OF A TOPOLOGICAL SPACE

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Recently, in connection with the study of systems of differential equations with multivalued and discontinuous right-hand sides ([1,2]), interest has again arisen in generalized dynamical systems ([3]). On the other hand, in connection with the creation by Soviet and American mathematicians ([4,5]) of the theory of general dynamical systems (R, G), where G is an arbitrary topological group, systems of multivalued mappings of a space have been considered ([6]) (also under very general assumptions). In addition, recently the theory of multivalued mappings of a topological space has begun to develop ([7,8]).

In this note a concept is proposed of a dynamical system without uniqueness as a semigroup of multivalued mappings of a space.

1. Let R be a topological space, S a topological semigroup (i.e. a topological space with a binary associative multiplicative operation, continuous in the aggregate of its components) with identity e; let f be a mapping assigning to each point $p \in R$ and each element $s \in S$ a nonempty bicompact set $f(p, s) \subset R$.

The collection (R, S, f) will be called a semigroup of multivalued mappings (s.m.m.) if the following three conditions are satisfied:

A_1. $f(p, e) = p$ for every point $p \in R$.

A_2. $f[f(p, s_1), s_2] = f(p, s_1 s_2)$ for any two elements s_1 and s_2 of S and any point $p \in R$.

A_3. If $f(p, s) = B$ $(p \in R,\ s \in S)$, then for any neighborhood $U(B)$ of the set B in R there exist neighborhoods $V(p)$ of the point p in R and $W(s)$ of the element s in S such that $f[V(p), W(s)] \subset U(B)$.

If $f(p, s)$ is a point for all $p \in R,\ s \in S$, then (R, S, f) will be called a semigroup of single-valued mappings (s.s.m.). The set

$$f(p, S) = \bigcup_{s \in S} f(p, s)$$

is called the funnel of the point p. Let $\{U_1, \dots, U_n\}$ be a finite system of open sets in R. Following ([8]), by

$$\langle U_1, \dots, U_n \rangle$$

we shall denote the collection of all closed subsets A of R such that

$$A \subseteq \bigcup_{i=1}^{n} U_i \quad \text{and} \quad A \cap U_i \neq \Lambda \quad (i = 1, \dots, n).$$

The collection of all $\langle U_1, \dots, U_n \rangle$ for arbitrary open sets $U_1, \dots, U_n$ forms a basis of the so-called finite ([8]) topology in the set of all closed subsets of the space R. We note that if the space R is metric, then the finite topology in the set of compact subsets of the space R is induced by the Hausdorff distance between two sets.

Theorem 1. *If R is a T_1-separated space, and $S = D^+$ (D^-), where D^+ (D^-) is the additive semigroup of nonnegative (nonpositive) real numbers with the usual topology, then for any $p \in R$, $s \in S$ and any open sets $U_1, \dots, U_n$ in R satisfying the condition*

$f(p, s) \in \langle U_1, \dots, U_n \rangle$, there is a neighborhood $W(s) \subset S$ such that $f(p, s') \in \langle U_1, \dots, U_n \rangle$ for all $s' \in W(s)$.

Let us note that if $S \neq D^+(D^-)$, then the assertion of Theorem 1 is, generally speaking, false.

If for the point $p \in R$ and arbitrary $s \in S$ the set $f(p, s)$ is connected, then the funnel $f(p, S)$ is connected. If $S = D^+(D^-)$, then, by Theorem 1 and the work ([7]), the funnel of any point $p \in R$ is connected.

2. We indicate the connection between generalized dynamical systems (g.d.s.) ([3,9]) and p.n.m. (R, D^+, f), where R is a metric space. A g.d.s. (R, D, f) determines two p.n.m. (R, D^+, f) and (R, D^-, f). Hence, by Theorem 1, it follows that axiom 5^0 of the work ([9]) is a consequence of the other axioms of that work.

Theorem 2. For (R, D^+, f), where R is a compact metric space and $f(R, t) = R$ for all $t \geqslant 0$, there exists a g.d.s. (R, D, f^*) such that $f^*(p, t) = f(p, t)$ for $t \geqslant 0$, $p \in R$.

If R is not compact, then the assertion of Theorem 2, generally speaking, loses its force.

Let us also note that if in axiom B of ([6]) the inclusion condition $f(p, g_1 g_2) \subseteq f[f(p, g_1), g_2]$ is replaced by the condition of equality for any elements g_1 and g_2 of the group G, then from this the single-valuedness of the mapping $f(p, g)$ for all $p \in R,\ g \in G$ will follow. Therefore it is natural to consider precisely semigroups of multivalued mappings.

3. Consider the set $\mathcal{C} = \mathcal{C}(R, S, f)$, whose elements are the sets $B = f(p, s)$ $(p \in R,\ s \in S)$, and endow $\mathcal{C}$ with the topology of a subspace of the space $\mathfrak{X}R$ ([7]): neighborhoods of a point $B \in \mathcal{C}$ will be the collections U' of all $B' \subseteq U(B)$, where U is an arbitrary neighborhood of the set B in $R,\ B' \in \mathcal{C}$. We denote the resulting space by $\mathcal{C}_1$. A p.n.m. (R, S, f) naturally induces on $\mathcal{C}_1$ a p.o.m. $(\mathcal{C}_1, S, F_1)$.

Let us now endow the set $\mathcal{C}$ with the finite ([8]) topology and denote the resulting space by $\mathcal{C}_2$. If $f(p, s) = B$ is a continuous mapping of $R \times S$ into $\mathcal{C}_2$, then the p.n.m. (R, S, f) induces, analogously to the preceding case, a p.o.m. $(\mathcal{C}_2, S, F_2)$.

We shall call p.n.m. (R, S, f) and (R', S', f') **isomorphic** if there exist a homeomorphic mapping φ of the space R onto R' and an isomorphic mapping φ^* of the topological semigroup S onto S' such that $\varphi[f(p, s)] = f'[\varphi(p), \varphi^*(s)]$ for all $p \in R,\ s \in S$. The consideration of p.o.m. $(\mathcal{C}_1, S, F_1)$ and $(\mathcal{C}_2, S, F_2)$ corresponding to the p.n.m. (R, S, f) is useful in many cases; however, there exist nonisomorphic p.n.m. (R, S, f) and (R', S', f') such that the p.o.m. $(\mathcal{C}_1, S, F_1)$ and $(\mathcal{C}'_1, S', F'_1)$ are isomorphic, and the p.o.m. $(\mathcal{C}_2, S, F_2)$ and $(\mathcal{C}'_2, S', F'_2)$ are isomorphic.

4. In what follows the topological space R is assumed to be Hausdorff. A set $A \subseteq R$ will be called **semi-invariant** (**quasi-invariant**) if $f(A, s) \subseteq A$ $(f(A, s) \supseteq A)$ for all $s \in S$. If $f(A, s) = A$ $(s \in S)$, then the set A is called **invariant**. A set A will be called **pseudo-invariant** if for every point $p \in A$ and every $s \in S$ one has $f(p, s) \cap A \neq \Lambda$. A set M is called **minimal with respect to a certain property** P if it is closed, nonempty, satisfies property P, and there is no proper closed subset $M' \subset M$ satisfying condition P.

Theorem 3. Every semi-invariant (quasi-invariant, invariant, pseudo-invariant) bicompact set contains a minimal semi-invariant (quasi-invariant, invariant, pseudo-invariant) bicompact set.

If the semigroup S is commutative, then a bicompact set is minimal semi-invariant if and only if it is minimal invariant.

Let us note that in a bicompact space the closure of a quasi-invariant set is quasi-invariant; if, however, the space is not bicompact, then this assertion is, generally speaking, false.

In what follows it is assumed that $S = D^+$. Introduce the concept of a motion $\varphi(p, t)$ $(t \geq 0)$ analogously to how this was done in ([9]), and denote

$$\varphi(p, D^+) = \bigcup_{t \geq 0} \varphi(p, t).$$

It is easy to prove the existence of a motion $\varphi(p, t)$ passing through an arbitrary point $q \in \varphi(p, D^+)$.

Theorem 4. *In order that a set A be pseudo-invariant, it is necessary and sufficient that for every point $q \in A$ there exist a motion $\varphi(q, t)$ such that $\varphi(q, D^+) \subseteq A$.*

In order that a bicompact set A be a minimal pseudo-invariant set, it is necessary and sufficient that for every point $q \in A$ and every motion $\varphi(q, t)$ such that $\varphi(q, D^+) \subseteq A$, one have

$$\overline{\varphi(q, D^+)} = A.$$

The notions of a bicompact minimal pseudo-invariant set and a bicompact minimal quasi-invariant set coincide. However, there exist pseudo-invariant sets that are neither semi-invariant nor quasi-invariant.

5. Let $p \in R$ and let E be an open or closed subset of R. Introduce the functions $(t \geq 0)$:

$$v_1(p, t, E) = \begin{cases} 1, & \text{if } f(p, t) \subseteq E, \\ 0, & \text{if } f(p, t) \cap (R \setminus E) \neq \Lambda; \end{cases}$$

$$v_2(p, t, E) = \begin{cases} 1, & \text{if } f(p, t) \cap E \neq \Lambda, \\ 0, & \text{if } f(p, t) \cap E = \Lambda. \end{cases}$$

We note that $v_2(p, t, E) = 1 - v_1(p, t, R \setminus E)$. From Theorem 1 and ([8]) it follows that the functions under consideration are measurable. Introduce the notation:

$$P_i(p, E) = \lim_{T \to \infty} \frac{1}{T} \int_0^T v_i(p, t, E)\, dt \quad (i = 1, 2);$$

if the indicated limits exist.

A bicompact set $Z_1(p)$ $(Z_2(p))$ will be called the **center of strong (weak) attraction** of the funnel of the point p, if for every neighborhood $U(Z_1)$ $(U(Z_2))$ one has

$$P_1[p, U(Z_1)] = 1 \quad (P_2[p, U(Z_2)] = 1).$$

Theorem 5. *If $\overline{f(p, D^+)}$ is bicompact, then there exists a unique minimal center of strong attraction $Z_1(p)$, and the set $Z_1(p)$ is quasi-invariant. Under the same condition there exists at least one minimal center of weak attraction $Z_2(p)$. All $Z_2(p)$ belong to $Z_1(p)$.*

Let $\{\varphi_\alpha(p,t)\}$ be the totality of all motions issuing from the point p. Introduce, as usual ([10]), the concept of the probability $P[\varphi_\alpha(p,t)]$ that the motion $\varphi_\alpha(p,t)$ is in the set E as $t \to +\infty$, and the concept of the center of attraction of a motion.

Theorem 6. *If $\overline{\varphi_\alpha(p, D^+)}$ is bicompact, then there exists a unique minimal center of attraction $Z[\varphi_\alpha(p,t)]$, and the set $Z[\varphi_\alpha(p,t)]$ is quasi-invariant.*

Let us note that $Z[\varphi_\alpha(p,t)]$ is a center of weak attraction of the funnel of the point p, but is not necessarily minimal. There exist $Z_2(p)$ that are not centers of attraction for any motion $\varphi_\alpha(p,t)$. All $Z[\varphi_\alpha(p,t)]$ belong to $Z_1(p)$.

The closure of the union of all $Z[\varphi_\alpha(p,t)]$, generally speaking, is a proper subset of the set $Z_1(p)$.

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Note: Figure translations are in progress. See original paper for figures.