L. D. FADDEEV

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Abstract

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MATHEMATICAL PHYSICS

L. D. FADDEEV

THE STRUCTURE OF THE RESOLVENT OF THE SCHRÖDINGER OPERATOR FOR A SYSTEM OF THREE PARTICLES AND THE SCATTERING PROBLEM

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In the present work a representation is obtained for the resolvent of the energy operator of a system of three particles in nonrelativistic quantum mechanics; with its aid the continuous spectrum of this operator is studied, the theorem of expansion in its eigenfunctions is proved, and the asymptotics, for large $|t|$, of the solutions of the corresponding nonstationary Schrödinger equation are investigated. The method of investigation is a development of the approach proposed in ([1]).

1. We introduce the basic notation and concepts. The operator H under consideration acts in the Hilbert space $\mathfrak{H} = \mathcal{L}_2(E_6)$ of square-integrable functions $f(k,p)$ of two three-dimensional vectors k and p. It is convenient for us to use three systems of coordinates in E_6: (k_{23}, p_1), (k_{31}, p_2), and (k_{12}, p_3), the relation between which follows from the relations

$$k_{23} = -p_2 - \frac{1}{2}p_1 = p_3 + \frac{1}{2}p_1 \tag{1}$$

and from those obtained from (1) by all possible cyclic permutations of the indices. In the case when it is immaterial to us which of the systems of variables is used, we shall write k and p without indices. To shorten formulas it is convenient to introduce an index α, taking the values $1,2,3$ or $23,31,12$, depending on the quantity with which it stands.

The operator H is defined as follows:

$$H = H_0 + V_{23} + V_{31} + V_{12}. \tag{2}$$

Here H_0 is the operator of multiplication by the function

$$k_{23}^2 + \frac{3}{4}p_1^2 = k_{31}^2 + \frac{3}{4}p_2^2 = k_{12}^2 + \frac{3}{4}p_3^2 \tag{3}$$

and the operators V_α ($\alpha = 23, 31, 12$) have kernels

$$V_\alpha(k,p;k',p') = v_\alpha(k_\alpha - k'_\alpha)\,\delta(p_\alpha - p'_\alpha). \tag{4}$$

With respect to the functions $v_\alpha(q)$ we require that the conditions

$$|v_\alpha(q)| \le K(1+|q|)^{-1-\varepsilon_0}, \qquad \varepsilon_0 > 0; \tag{5}$$

$$|v_\alpha(q) - v_\alpha(q+h)| \le K(1+|q|)^{-1-\varepsilon_0}|h|^{\mu_0}, \qquad \mu_0 > \frac{1}{2}. \tag{6}$$

be satisfied.

The main result of the work ([1]) may be formulated as follows: let

$$R(z) = (H - zI)^{-1}; \qquad R_\alpha(z) = (H_\alpha - zI)^{-1}; \qquad H_\alpha = H_0 + V_\alpha. \tag{7}$$

Under condition (5), for $R(z)$ the representation

$$R(z) = R_0(z) + \sum_\alpha (R_\alpha(z) - R_0(z)) + R_0(z)\left(\sum_{\alpha,\beta} W_{\alpha\beta}(z)\right) R_0(z), \tag{8}$$

is valid.

where $W_{\alpha\beta}(z)$ are integral kernel operators for which, when $\operatorname{Im} z \ne 0$, the estimates hold:

$$\left|W_{\alpha\beta}(k,p;k',p';z)\right| \le KM(k,p;k',p';\varepsilon)(1 + p_\alpha^2 + p'^2_\beta)^{-1}. \tag{9}$$

Here $\varepsilon < \varepsilon_0$ and

$$M(k,p;k',p';\varepsilon) = \sum_{\substack{\alpha,\beta\\ \alpha\ne\beta}} \left\{(1 + |p_\alpha - p'_\alpha|)(1 + |p_\beta - p'_\beta|)\right\}^{-1-\varepsilon}. \tag{10}$$

2. The assertion formulated above is obtained in (1) on the basis of an investigation of the system of integral equations for $W_{\alpha\beta}(z)$:

$$W_{\alpha\beta}(z) = W^{(0)}_{\alpha\beta}(z) - T_\alpha(z) R_0(z) \sum_{\gamma\ne\alpha} W_{\alpha\beta}(z). \tag{11}$$

Here we use a concise notation for integral operators. The kernel $T_\alpha(z)$ has the form

$$T_\alpha(k,p;k',p';z) = t_\alpha(k_\alpha, k'_\alpha, z - \tfrac{3}{4}p_\alpha^2)\,\delta(p_\alpha - p'_\alpha). \tag{12}$$

The free term is defined as follows:

$$W_{\alpha\alpha}^{(0)} = 0; \qquad W_{\alpha\beta}^{(0)}(z) = T_\alpha(z) R_0(z) T_\beta(z), \quad \alpha \neq \beta. \tag{13}$$

The kernels $t_\alpha(k, k', z)$ are solutions of the equations

$$t_\alpha(k, k', z) = v_\alpha(k - k') - \int v_\alpha(k - k'')(k''^2 - z) t_\alpha(k'', k', z)\, dk'', \tag{14}$$

which one has to deal with in studying the operators

$$h_\alpha f(k) = k^2 f(k) + \int v_\alpha(k - k') f(k')\, dk', \tag{15}$$

acting in $\mathcal{L}_2(E_3)$ and corresponding to the two-particle problem. Under conditions (5) and (6) on $v_\alpha(q)$, the operators h_α have a continuous spectrum on the positive part of the real axis and a finite number of negative eigenvalues of finite multiplicity. To simplify the notation of the formulas we shall assume that each h_α has one simple eigenvalue at the point $-\chi_\alpha^2 < 0$; the corresponding normalized eigenfunctions will be denoted by $\psi_\alpha(k)$. In this case, for the kernels $t_\alpha(k, k', z)$ the representation is valid

$$t_\alpha(k, k', z) = \varphi_\alpha(k) \overline{\varphi_\alpha(k')} (z + \chi_\alpha^2)^{-1} + \tilde{t}_\alpha(k, k', z), \tag{16}$$

where $\varphi_\alpha(k) = (k^2 + \chi_\alpha^2)\psi_\alpha(k)$. For $\varphi_\alpha(k)$ and $\tilde{t}_\alpha(k, k', z)$ there hold estimates analogous to estimates (5) and (6) for $v_\alpha(k)$ and $v_\alpha(k - k')$, respectively, with exponents $\varepsilon < \varepsilon_0$ and $\mu < \mu_0$. The estimates for $\tilde{t}_\alpha(k, k', z)$ are uniform in z over the entire complex plane with a cut along the positive part of the real axis. The limiting values $\tilde{t}_\alpha(k, k', z)$ on the cut satisfy a Hölder condition in z with exponent $\mu = 1/2$.

Iteration of system (11) tells us the character of the singularities of the kernels $W_{\alpha\beta}(k, p; k', p'; z)$ in a neighborhood of the real axis. Let $W_{\alpha\beta}^{(n)}(z)$ denote the result of the n-th iteration. By means of reducing system (11) to

operator equation with a completely continuous operator in a suitably chosen Banach space, and on the basis of the study of the corresponding homogeneous equation, the following result is obtained:

Theorem 1. For the kernels $Q_{\alpha\beta}(z) = W_{\alpha\beta}(z) - \sum_{n=0}^{2} W_{\alpha\beta}^{(n)}(z)$ the representation

$$\begin{aligned} Q_{\alpha\beta}(k, p; k', p'; z) = {} & \overset{\circ\circ}{Q}_{\alpha\beta}(k, p; k', p'; z) \\ & + \frac{\varphi_\alpha(k_\alpha)}{z + \varkappa_\alpha^2 - \frac{3}{4}p_\alpha^2} \overset{\circ *}{Q}_{\alpha\beta}(p_\alpha; k', p'; z) + \overset{*\circ}{Q}_{\alpha\beta}(k, p; p'_\beta; z) \frac{\overline{\varphi_\beta(k'_\beta)}}{z + \varkappa_\beta^2 - \frac{3}{4}p'^2_\beta} \\ & + \frac{\varphi_\alpha(k_\alpha)}{z + \varkappa_\alpha^2 - \frac{3}{4}p_\alpha^2} \overset{**}{Q}_{\alpha\beta}(p_\alpha; p'_\beta; z) \frac{\overline{\varphi_\beta(k'_\beta)}}{z + \varkappa_\beta^2 - \frac{3}{4}p'^2_\beta}. \end{aligned} \tag{17}$$

holds.

The kernels $\overset{\circ\circ}{Q}_{\alpha\beta}(z)$ satisfy the estimates (9), and in order to obtain estimates for the kernels $\overset{\circ *}{Q}_{\alpha\beta}(z)$, $\overset{*\circ}{Q}_{\alpha\beta}(z)$, or $\overset{**}{Q}_{\alpha\beta}(z)$, one must put, respectively, $k_\alpha = 0$, $k'_\beta = 0$, or simultaneously $k_\alpha = 0$ and $k'_\beta = 0$ in the right-hand side of (9). Analogous estimates hold for the Faddeev differences with exponent $\mu < 1/4$ in all variables on which these kernels depend. The estimates described are uniform in z in any finite domain of the complex plane with a cut along the positive part of the real axis, extending to ∞ from the point $\lambda_0 = -\max(\varkappa_{23}^2, \varkappa_{31}^2, \varkappa_{12}^2)$, except for neighborhoods of the real points λ_n, $n = 1, \dots$, which are points of the discrete spectrum of the operator H. The points λ_n are concentrated on a finite interval and can have as accumulation points only the points $-\varkappa_\alpha^2$ $(\alpha = 23, 31, 12)$.

A representation of the type (17) is naturally written also for the kernels $W_{\alpha\beta}(z)$ themselves. We shall denote the corresponding kernels by $\overset{\circ\circ}{W}_{\alpha\beta}$, $\overset{\circ *}{W}_{\alpha\beta}$, $\overset{*\circ}{W}_{\alpha\beta}$, $\overset{**}{W}_{\alpha\beta}$. These kernels contain terms arising from the first three iterations, for which estimates of the type (9) are not valid. However, the singularities of these terms can be investigated separately.

3. With the aid of the behavior of the resolvent of the operator H studied above, we can completely characterize its spectrum. Apart from the discrete spectrum at the points λ_n $(n = 1, \dots)$, H has an absolutely continuous spectrum on the half-axis $[\lambda_0, \infty)$. We shall give a more detailed characterization of this spectrum. Let $\mathfrak{H}_0 = \mathfrak{H}$ and let $\mathfrak{H}_\alpha$ $(\alpha = 23, 31, 12)$ be three copies of the space $L_2(E_3)$. Let D_0, D_α be dense sets in $\mathfrak{H}_0$ and $\mathfrak{H}_\alpha$, consisting of functions $f_0(k,p)$ and $f_\alpha(p)$, respectively, finite, sufficiently smooth, and vanishing in neighborhoods of the surfaces $k^2 + \frac{3}{4}p^2 = \lambda_n$, $-\varkappa_\alpha^2 + \frac{3}{4}p_\alpha^2 = \lambda_n$, respectively. For $f_0 \in D_0$ and $f_\alpha \in D_\alpha$ the following expressions have meaning:

$$U_0^{(\pm)} f_0(k,p) = f_0(k,p) - \iint \left\{ \sum_\alpha T_\alpha(k,p;k',p';\omega'_0 \mp i0) - \sum_{\alpha,\beta} W_{\alpha\beta}(k,p;k',p';\omega'_0 \mp i0) \right\} \left(k^2 + \frac{3}{4}p^2 - \omega'_0 \mp i0 \right)^{-1} f_0(k',p')\, dk'\, dp', \tag{18}$$

$$U_\alpha^{(\pm)} f_\alpha(p) = \psi_\alpha(k_\alpha) f_\alpha(p_\alpha) - \int \left(\sum_\beta \left\{ \overset{\circ *}{W}_{\beta\alpha}(k,p;p'_\alpha;\omega'_\alpha \mp i0) \right.\right.$$
$$\left.\left. + \varphi_\beta(k_\beta) \left(\varkappa_\beta^2 - \frac{3}{4}p_\beta^2 + \omega'_\alpha \mp i0 \right)^{-1} \overset{**}{W}_{\beta\alpha}(p_\beta;p'_\alpha;\omega'_\alpha \mp i0) \right\} \right)$$
$$\times \left(k^2 + \frac{3}{4}p^2 - \omega'_\alpha \mp i0 \right)^{-1} f_\alpha(p'_\alpha)\, dp'_\alpha; \tag{19}$$

$$\omega_0 = \frac{3}{4}p^2 + k^2; \qquad \omega_\alpha = \frac{3}{4}p_\alpha^2 - \varkappa_\alpha^2.$$

Theorem 2. The operators $U_0^{(\pm)}$ and $U_\alpha^{(\pm)}$ extend, upon closure, to isometric operators acting from $\mathfrak{H}_0$ into $\mathfrak{H}$ or from $\mathfrak{H}_\alpha$ into $\mathfrak{H}$, respectively. The following assertions hold:

I.
$$\exp\{iHt\}\, U_0^{(\pm)} f_0(k,p) = U_0^{(\pm)}(\exp\{i\omega_0 t\} f_0(k,p)),$$
$$\exp\{iHt\}\, U_\alpha^{(\pm)} f_\alpha(p_\alpha) = U_\alpha^{(\pm)}(\exp\{i\omega_\alpha t\} f_\alpha(p_\alpha)).$$

II.
$$(U_\alpha^{(\pm)} f_\alpha,\, U_\beta^{(\pm)} g_\beta)_{\mathfrak{H}} = (f_\alpha, g_\alpha)_{\mathfrak{H}_\alpha}\, \delta_{\alpha\beta}, \quad \alpha = 0,\, 23,\, 31,\, 12.$$

III. For any $f \in \mathfrak{H}$ orthogonal to all eigenfunctions of the discrete spectrum of H, there are uniquely determined $f_0^{(\pm)} \in \mathfrak{H}_0$ and $f_\alpha^{(\pm)} \in \mathfrak{H}_\alpha$ ($\alpha = 23, 31, 12$) such that

$$f = U_0^{(\pm)} f_0^{(\pm)} + U_{23}^{(\pm)} f_{23}^{(\pm)} + U_{31}^{(\pm)} f_{31}^{(\pm)} + U_{12}^{(\pm)} f_{12}^{(\pm)}.$$

Assertion I permits one to call the kernels of the operators $U_0^{(\pm)}$ and $U_\alpha^{(\pm)}$ the eigenfunctions of the continuous spectrum of the operator with eigenvalues ω_0 and ω_α, respectively. Assertion II contains the orthogonality and normalization conditions, and III—the completeness of these functions.

4. The results obtained make it possible to study the asymptotics, for large $|t|$, of the operator $\exp\{iHt\}$.

Let P_α be the projector in $\mathfrak{H}$ onto the subspace formed by functions of the form $\psi_\alpha(k_\alpha) f(p_\alpha)$, where $f(p)$ is arbitrary.

Theorem 3. There exist strong limits

$$\widetilde{U}_0^{(\pm)} = \lim_{t\to\pm\infty} e^{iHt} e^{-iH_0 t}; \qquad \widetilde{U}_\alpha^{(\pm)} = \lim_{t\to\pm\infty} e^{iHt} e^{-iH_\alpha t} P_\alpha.$$

Moreover,

$$\widetilde{U}_0^{(\pm)} f = U_0^{(\pm)} f; \qquad \widetilde{U}_\alpha^{(\pm)}(\psi_\alpha f_\alpha) = U_\alpha^{(\pm)} f_\alpha. \tag{20}$$

In the direct sum $\mathcal{E} = \mathfrak{H}_0 \oplus \mathfrak{H}_{23} \oplus \mathfrak{H}_{31} \oplus \mathfrak{H}_{12}$ one may consider the operator S, defined by the matrix

$$S_{\alpha\beta} = U_\alpha^{(+)*} U_\beta^{(-)} \qquad (\alpha, \beta = 0,\ 23,\ 31,\ 12). \tag{21}$$

It follows from Theorem 2 that the operator S is unitary in $\mathcal{E}$. The result of Theorem 3 permits one to call it the scattering operator for the system described by the operator H. It is not difficult to establish a connection between the kernels of the operators $S_{\alpha\beta}$ and the kernels $W_{\alpha\beta}(z)$. For lack of space we do not present the corresponding formulas.

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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1. L. D. Faddeev, DAN, 138, No. 3, 565 (1961).

Note: Figure translations are in progress. See original paper for figures.