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Abstract

Full Text

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LINEAR POLYNOMIAL OPERATIONS

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1.

Let $\widetilde{C}$ be the space of all continuous 2π-periodic functions $f(x)$ with norm

$$\|f\|_{\widetilde{C}} = \max_{0\le x<2\pi} |f(x)|;$$

let p and q be integers satisfying the inequalities $0 \le q \le p$. By $\Delta_{q,p}$ we shall denote the set of all linear operators $U_{q,p}$ from $\widetilde{C}$ into $\widetilde{C}$ satisfying the following conditions: 1) for any $f \in \widetilde{C}$, $U_{q,p}(f,x)$ is a trigonometric polynomial of order $\le p$; 2) if $f(x)$ is a trigonometric polynomial of order $\le q$, then $U_{q,p}(f,x) = f(x)$. Put

$$\rho_{q,p} = \inf_{U_{q,p}\in\Delta_{q,p}} \|U_{q,p}\|.$$

Since $\Delta_{0,p} \supset \Delta_{1,p} \supset \cdots \supset \Delta_{p,p}$, we have

$$\rho_{0,p} \le \rho_{1,p} \le \cdots \le \rho_{p-1,p} \le \rho_{p,p}.$$

There arises the natural question of finding, in each class of operators $\Delta_{q,p}$, $q = 0, 1, 2, \dots, p$, an operator of least norm and of computing $\rho_{q,p}$. The solution of this question is given by Theorem 1.

Theorem 1. *In the class of operators $\Delta_{q,p}$, the operator of least norm is*

$$\overline{U}_{q,p}(f,x) = \frac{1}{\pi}\int_0^{2\pi} f(x+t)\left[D_q(t) + \sum_{i=1}^{p-q} \tilde{\alpha}_i \cos(q+i)t\right] dt,$$

where $D_q(t)$ is the Dirichlet kernel of order q, and the numbers $\{\tilde{\alpha}_i\}_{i=1}^{p-q}$ are determined from the condition that the integral

$$I(\alpha_1, \alpha_2, \dots, \alpha_{p-q}) = \int_0^{\pi}\left|D_q(t) + \sum_{i=1}^{p-q} \alpha_i \cos(q+i)t\right| dt$$

be minimal. Moreover,

$$\rho_{q,p} = \frac{2}{\pi}\int_0^{\pi}\left|D_q(t) + \sum_{i=1}^{p-q} \tilde{\alpha}_i \cos(q+i)t\right| dt.$$

This theorem is a consequence of the general Theorem 2 from (^{1}). With the aid of Theorem 1 it is easy to obtain the following results.

A. In the class of operators $\Delta_{0,p}$, the partial Fejér sum of order p has least norm, and

$$\rho_{0,p} = 1.$$

B. In the class of operators $\Delta_{p-1,p}$, the operator

$$\overline{U}_{p-1,p} = \frac{1}{2\pi} \int_0^{2\pi} f(x+t) \sin pt \, \cot \frac{t}{2} \, dt \tag{1}$$

has least norm, and

$$\rho_{p-1,p} = \frac{1}{2\pi} \int_0^{2\pi} \left| \sin pt \, \cot \frac{t}{2} \right| dt \simeq \frac{4}{\pi^2} \ln p.$$

The operator (1) often occurs in the theory of Fourier series (see, for example, (^{2,3})), but its extremal property apparently has not been noted until now.

C. In the class $\Delta_{p,p}$, the partial sum of the series has least norm ⋯

Fourier sum $S_p(f,x)$ and $\rho_{p,p} = L_p$, where L_p is the Lebesgue constant of order p. This result is known (^{4}).

Let us note that assertion A can also easily be obtained without Theorem 1.

2. For given q and p one can construct the de la Vallée-Poussin partial sum

$$\sigma_{q,p} = \sigma_{q,p}(f,x) = \frac{S_q(f,x) + S_{q+1}(f,x) + \ldots + S_p(f,x)}{p-q+1}.$$

It is obvious that $\sigma_{q,p} \in \Delta_{q,p}$. The question arises: when is the partial sum $\sigma_{q,p}$ extremal in the class of operators $\Delta_{q,p}$?

Theorem 1 makes it possible to answer this question as well.

Theorem 2. *In order that $\sigma_{q,p}$ be an extremal operator in the class $\Delta_{q,p}$, it is necessary and sufficient that $2q$ be divisible by $(p-q+1)$.*

Since $2q$ is divisible by $(p-q+1)$ for q respectively equal to $0, (p-1), p$, it follows, according to Theorem 2, that $\sigma_{0,p}, \sigma_{p-1,p}, \sigma_{p,p}$ are extremal operators respectively for the classes $\Delta_{0,p}, \Delta_{p-1,p}, \Delta_{p,p}$. It is not difficult to see that we have again obtained assertions A, B, C, since $\sigma_{0,p}, \sigma_{p-1,p}, \sigma_{p,p}$ coincide respectively with the Fejér partial sum of order p, with the operator (1), and with $S_p(f,x)$. In this connection the result of Theorem 3 is of interest.

Theorem 3. *If $(p+1)$ is a prime number, then among the partial sums $\sigma_{0,p}, \sigma_{1,p}, \dots, \sigma_{p-1,p}, \sigma_{p,p}$, only the partial sums $\sigma_{0,p}, \sigma_{p-1,p}, \sigma_{p,p}$ are extremal operators respectively in their classes $\Delta_{0,p}, \Delta_{p-1,p}, \Delta_{p,p}$. Every other de la Vallée-Poussin partial sum is not an extremal operator in its class.*

3. Let us consider one more problem which can be solved with the help of Theorem 1. Let $\overline{\Delta}_{q,2p}$ be the set of all linear operators $W_{q,2p}(f,x)$ from $\widetilde{C}$ into $\widetilde{C}$ possessing the following properties: 1) for every $f \in \widetilde{C}$, $W_{q,2p}(f,x)$ is a trigonometric polynomial of order $\leq p$; 2) if $f \in \widetilde{C}$ is a polynomial of order $\leq q$, then $W_{q,2p}(f,x) = \overline{f}(x)$, where $\overline{f}(x)$ is the polynomial conjugate to $f(x)$. Put

$$\overline{\rho}_{q,2p} = \inf_{W \in \overline{\Delta}} \|W_{q,1p}\|.$$

Theorem 4. *In the class $\overline{\Delta}_{2p-1,2p}$ the operator*

$$W_{2p-1,2p}(f,x) = \frac{1}{\pi} \int_0^{2\pi} f(x+t)\Phi(t)\,dt,$$

has the smallest norm, where

$$\Phi(t) = -\left(\frac{\sin 2pt}{2} + \sum_{k=1}^{2p-1} \sin kt \right);$$

$$\overline{\rho}_{2p-1,2p} = \frac{2}{\pi} \left(\frac{1}{1} + \frac{1}{3} + \ldots + \frac{1}{2p-1} \right).$$

4. As is known (^{5,6}), the general definition of a linear trigonometric polynomial operation of type Φ_n is as follows: 1) $U_n(f,x)$ is a linear operator from $\widetilde{C}$ into $\widetilde{C}$; 2) for every $f \in \widetilde{C}$, $U_n(f,x)$ is a trigonometric polynomial of order not exceeding n; 3) for every trigonometric polynomial $f(x)$ of order not exceeding n the equality

$$U_n(f,x) = \sigma_n(f,x),$$

holds, where

$$\sigma_n(f,x) = \int_0^{2\pi} f(x,t)\Phi_n(t)\,dt \tag{*}$$

and $\Phi_n(t)$ is a given trigonometric polynomial of order n.

Suppose that for every $f \in \widetilde{C}$ the relation

$$\sigma_n(f,x) \to f(x), \qquad n \to \infty. \tag{2}$$

holds uniformly.

The question is whether, starting from an arbitrary operation $U_n(f, x)$ of type Φ_n, one can recover the function $f(x)$ as $n \to \infty$. The answer to this question is positive in principle, for Theorem 5 holds.

Theorem 5. *Let, for every $f \in \widetilde{C}$, the relation (2) hold uniformly. Then for every $f(x)$, every x, and every operation $U_n(f, x)$ of type Φ_n, there exists a real number τ_n, depending on f, x, n, such that*

$$\lim_{n\to\infty} U_n(f_{\tau_n}, x - \tau_n) = f(x),$$

where $f_t(x) = f(x + t)$.

The proof of the theorem follows from the equality (5) *

$$\frac{1}{2\pi}\int_0^{2\pi} U_n(f_t, x - t)\, dt = \sigma_n(f, x), \tag{3}$$

if the mean-value theorem is applied to its left-hand side. Here one must take into account that $U_n(f_t, x - t)$ is a continuous function of t.

5. At present there exist two methods for studying polynomial operations. The starting point of one method is Faber’ s work (^{7}), and the starting point of the other method is Fejér’ s work (^{8}). The second method is well known, since it is presented in many textbooks ($^{9-12}$). In this connection it is curious that the second method is, in a certain sense, a consequence of the first method. The basis of the first method is formula (3) and its special case

$$\frac{1}{2\pi}\int_0^{2\pi} U_n(f_t, x - t)\, dt = S_n(f, x), \tag{4}$$

which holds if, for every polynomial T of order n, $U_n(T, x) = T(x)$. Applying the mean-value theorem to the left-hand side of (4), we obtain that there exists a real α such that

$$U_n(f_\alpha, x - \alpha) = S_n(f, x).$$

In particular, there exists an α such that

$$U_n(f_\alpha, -\alpha) = S_n(f, 0).$$

This equality and its various modifications constitute the basis of the second method.

*

- Incidentally, this equality generalizes to the case where the convolution (*) is replaced by the integral

$$\sigma(f, x) = \int_0^{2\pi} f(t)\Phi_{n,m}(x, t)\, dt,$$

where $\Phi_{n,m}(x, t)$ is a trigonometric polynomial of order n with respect to x and of order m with respect to t, with $m \leq n$. In this case equality (3) takes the form:

$$\frac{1}{2\pi}\int_0^{2\pi} U_n(f_t, x - t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} f(z)\, dz \int_0^{2\pi} \Phi_{n,m}(x - t, z - t)\, dt.$$

6. Let

$$0 \leqslant x_0^{(n)} < x_1^{(n)} < \cdots < x_{2n}^{(n)} \leqslant 2\pi, \qquad (m_1)$$

$$0 \leqslant y_0^{(n)} < y_1^{(n)} < \cdots < y_{2n}^{(n)} \leqslant 2\pi \qquad (m_2)$$

be two arbitrary systems of nodes of trigonometric interpolation. By $L_n(f, x, m_i)$, $i = 1, 2$, denote the corresponding Lagrange interpolation polynomials of order n. Marcinkiewicz [13] proved the equality

$$\frac{1}{2\pi}\int_0^{2\pi} L_n(f_t, x - t, m_i)\, dt = S_n(f, x); \qquad i = 1, 2$$

(which is a special case of (4)). Therefore

$$\frac{1}{2\pi}\int_0^{2\pi} [L_n(f_t, x - t, m_1) - L_n(f_t, x - t, m_2)]\, dt = 0.$$

Since the integrand depends continuously on t, by the mean-value theorem there exists a real a, $a = a(f, x, m_1, m_2)$, such that

$$L_n(f_a, x - a, m_1) = L_n(f_a, x - a, m_2). \qquad (5)$$

It is not difficult to verify that

$$L_n(f_a, x - a, m_i) = L_n(f, x, m_i + a), \qquad i = 1, 2,$$

where by $m_i + a$ is denoted the system of nodes obtained from the system m_i by a shift by a. Therefore equality (5) takes the form

$$L_n(f, x, m_1 + a) = L_n(f, x, m_2 + a).$$

Thus the following theorem has been proved:

Theorem 6. *Let* m_i*,* $i = 1, 2$*, be arbitrary systems of interpolation nodes and* $f \in \widetilde{C}$*. Then for any point* x *one can specify a shift of the nodes* m_1 *and* m_2 *such that the values at the point* x *of the Lagrange interpolation polynomials constructed for the shifted nodes are equal to each other.*

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Note: Figure translations are in progress. See original paper for figures.