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Abstract

Full Text

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ON A PROBLEM OF S. L. SOBOLEV

(Presented by Academician S. L. Sobolev on 25 VI 1962)

Consider the equation:

$$\sum_{i=0}^N \frac{\partial^i}{\partial t^i} \left(a_{i,1} \frac{\partial^2 u}{\partial x^2} + a_{i,2} \frac{\partial^2 u}{\partial y^2} + b_i(x, y) \frac{\partial u}{\partial x} + c_i(x, y) \frac{\partial u}{\partial y} + d_i(x, y) u \right) = 0, \quad (1)$$

where $a_{i,1}, a_{i,2}$ are constant real numbers; b_i, c_i, d_i are continuous functions of their arguments in a bounded, closed domain Ω with thrice differentiable boundary Γ , whose curvature is positive. In ⁽²⁾ the question of the almost-periodicity of solutions of the equation

$$\frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial^2 u}{\partial y^2} = 0 \quad (2)$$

was studied under the condition

$$u|_{\Gamma} = 0. \quad (3)$$

This problem is connected with the study of the spectrum of the singular integral operator A :

$$A = \Delta^{-1} \frac{\partial^2}{\partial y^2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Delta^{-1} u = u, \quad \Delta^{-1} u|_{\Gamma} = 0.$$

In ⁽²⁾ the dependence of the character of the spectrum A on the topological properties of Γ is shown. In ⁽³⁾ a certain class of solutions of equation (2), satisfying condition (3), is constructed. Using their properties, in ⁽⁴⁾ invariant subspaces of the operator A are constructed, in which a simple representation of this operator is given. Using this representation, it is easy to construct the spectral function and the resolvent for A in a certain infinite-dimensional Hilbert space $H \subset \overset{0}{W}_2^1(\Omega)$. We shall construct solutions of equation (1), satisfying condition (3) and being a generalization of the solutions constructed in ⁽³⁾. These solutions are not identically zero if the operator A , constructed for the domain Ω , does not have a purely point spectrum.

Let

$$(-1)^{i+1} \mu_i \frac{\partial \mu_i}{\partial x} = \frac{\partial \mu_i}{\partial y}, \quad \mu_1|_{\Gamma} = \mu_2|_{\Gamma} = \rho(\varphi). \quad (4)$$

The existence of such functions was established in (3).

Introduce the notation:

$$P(\lambda) = \sum a_{i,1} \lambda^i, \quad Q(\lambda) = \sum a_{i,2} \lambda^i, \quad \sum b_i(x, y) \lambda^i = B(\lambda),$$

$$\sum c_i \lambda^i = C(\lambda), \quad \sum d_i \lambda^i = D(\lambda). \quad (5)$$

Let $\lambda(\alpha)$ be a solution of the equation

$$P(\lambda) + \alpha^2 Q(\lambda) = 0. \quad (6)$$

We shall construct solutions of equation (1) of the form

$$u(x, y, t) = \int_{\mu_1}^{\mu_2} f(x, y, \alpha) e^{\lambda(\alpha)t} d\alpha. \quad (7)$$

The functions (7), obviously, satisfy condition (3). Using (4) and the equalities obtained from (4) by differentiation, for $f(x, y, \alpha)$ one may obtain the equations:

$$\frac{\partial \mu_2}{\partial x} \left\{ \alpha \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} - \frac{B - \alpha C}{2Q(\lambda)\alpha} f \right\}_{\alpha=\mu_2} = 0; \quad (8)$$

$$\frac{\partial \mu_1}{\partial x} \left\{ -\alpha \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{B + \alpha C}{2Q(\lambda)\alpha} f \right\}_{\alpha=\mu_1} = 0; \quad (9)$$

$$-\alpha^2 \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{B}{Q} \frac{\partial f}{\partial x} + \frac{C}{Q} \frac{\partial f}{\partial y} + \frac{D}{Q} f = 0. \quad (10)$$

If it is assumed that $Q(\lambda(\alpha)) \neq 0$, i.e., that P, Q have no common zeros, then the coefficients of equations (8), (9), (10) will be continuous functions of x and y . As in (4), the domain Ω can be divided into domains $\Pi_{i,j}$ such that in each of them, for the functions

$$F_i(x, y, \alpha) = x + \alpha y - A_i(\alpha), \quad R_j(x, y, \alpha) = x - \alpha y - B_j(\alpha) \quad (11)$$

the identities are fulfilled

$$F_i(x, y, \mu_1) \equiv 0, \quad R_j(x, y, \mu_2) \equiv 0. \quad (12)$$

Among the $\Pi_{i,j}$ there may be domains in which $\partial\mu_1/\partial x \equiv 0$, or $\partial\mu_2/\partial x \equiv 0$, or both of these identities hold simultaneously. Then the corresponding identities (12) do not occur, but equations (8), (9) are fulfilled by virtue of $\partial\mu_i/\partial x \equiv 0$.

Thus, in order to satisfy equations (8), (9), (10), it suffices to find a solution $f(x, y, \alpha)$ of equation (10) such that for $x, y \in \Pi_{i,j}$ the conditions

$$\alpha \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} - \frac{B - \alpha C}{2Q(\lambda)\alpha} f \Big|_{x - \alpha y - B_j(\alpha) = 0} = 0; \quad (13)$$

$$-\alpha \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{B + \alpha C}{2Q(\lambda)\alpha} f \Big|_{x + \alpha y - A_i(\alpha) = 1} = 0. \quad (14)$$

The existence of a solution of problem (13), (14) for equation (10), as well as its differentiability with respect to the parameter α , is easily proved by using the method of successive approximations. Multiplication of solutions of problem (13), (14) for equation (10) by a function depending only on α again leads to a solution of this problem. Thus we have constructed solutions of equation (1), under condition (3), having the form (7). The first derivatives with respect to x and y of the functions (7) differ from functions almost periodic in t , for fixed x, y , by a term vanishing as $t \rightarrow \infty$, if $\text{Re } \lambda(\alpha) \equiv 0$.

In the general case the functions (7) have discontinuities on the characteristics of equation (4) that are the boundaries of the domains $\Pi_{i,j}$. In some cases $f(x, y, \alpha)$ are continuous. Let us give a few examples. Let Ω be a domain whose boundary equation Γ has the form

$$x = \frac{\sin \varphi + \cos \varphi}{2}, \quad y = \frac{\cos \varphi - \sin \varphi}{2(1 + \cos^2 \varphi)}. \quad (15)$$

Then the functions

$$\mu_1 = \frac{1 - 2xy - \sqrt{1 - 4xy - 4y^2}}{2y^2}, \quad \mu_2 = \frac{|2xy - 1 + \sqrt{1 - 4xy + 8y^2}|}{2y^2} \quad (16)$$

are defined in Ω , satisfy equations (4), and

$$\mu_1|_{\Gamma} = \mu_2|_{\Gamma} = 1 + \cos^2 \varphi.$$

Example 1. For the equation

$$\frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - n^2 u \right) + \frac{\partial^2 u}{\partial y^2} = 0, \quad (17)$$

$$f(x, y, \alpha) = J_0 \left(n \sqrt{-F_1 F_2} \right),$$

where J_0 is the Bessel function,

$$F_1(x, y, \alpha) = \begin{cases} x + \alpha y + \sqrt{\alpha - 1}, & x + y < 0, \\ x + \alpha y - \sqrt{\alpha - 1}, & x + y > 0; \end{cases}$$

$$F_2(x, y, \alpha) = \begin{cases} x - \alpha y + \sqrt{2 - \alpha}, & 2y - x > 0, \\ x - \alpha y - \sqrt{2 - \alpha}, & 2y - x < 0; \end{cases}$$

$f(x, y, \alpha)$ is discontinuous on the straight lines $x = 2y$, $y = -x$.

Equation (17) is obtained from the equation considered in (1),

$$\frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \frac{\partial^2 u}{\partial y^2} = 0, \quad (18)$$

and the solution of equation (18), vanishing on the boundary of a certain cylinder in the space x, y, z , has the form:

$$u(x, y, z, t) = \sin nz \int_{\mu_1}^{\mu_2} J_0 \left(n \sqrt{-F_1 F_2} \right) f(\alpha) \exp \left[\pm i \sqrt{\frac{\alpha^2}{1 + \alpha^2}} t \right] d\alpha.$$

Example 2. The equation

$$\frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_4^2} \right) + \frac{\partial^2 u}{\partial x_4^2} = 0 \quad (19)$$

admits solutions

$$u = v(x, y, t); \quad x = \sqrt{x_1^2 + x_2^2 + x_3^2}; \quad x_4 = y;$$

$$\frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 v}{\partial x^2} + \frac{2}{x} \frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{\partial^2 v}{\partial y^2} = 0. \quad (20)$$

In the case of equation (20) we have:

$$f(x, y, \alpha) = \frac{g(\alpha)}{x},$$

and one can obtain a solution of equation (19), vanishing on the boundary of a certain torus in the space x_1, x_2, x_3, x_4 and not continuously differentiable.

Example 3. For the equation

$$\frac{\partial^2}{\partial t^2} \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - (x^2 + y^2)u \right] + \frac{\partial^2 u}{\partial y^2} - x^2 u = 0 \quad (21)$$

we have

$$F_1(x, y, \alpha) = (x + \alpha y)^2 - (\alpha - 1),$$

$$F_2(x, y, \alpha) = (x - \alpha y)^2 - (2 - \alpha),$$

$$f(x, y, \alpha) = J_0 \left(\frac{1}{2\alpha} \sqrt{F_1 F_2} \right). \quad (22)$$

The solution of equation (21), constructed with the aid of the function (22), is continuously differentiable in Ω .

In conclusion we note that, by separation of variables, equation (18) can be reduced to equations having the form (1) in the case where Ω is a domain of revolution about the y -axis in the space x, y, z .

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Note: Figure translations are in progress. See original paper for figures.

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