CONSTRUCTION OF A NUMBER \(\alpha\) FOR WHICH THE FRACTIONAL PARTS \(\{\alpha g^x\}\) ARE RAPIDLY UNIFORMLY DISTRIBUTED

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Abstract

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CONSTRUCTION OF A NUMBER α FOR WHICH THE FRACTIONAL PARTS $\{\alpha g^x\}$ ARE RAPIDLY UNIFORMLY DISTRIBUTED

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Let $g \geq 2$ be an integer, α a real number, $\mathfrak{M}$ an interval lying on the half-segment $[0, 1)$, $\operatorname{mes} \mathfrak{M}$ its length, and $N_P(\mathfrak{M})$ the number of numbers $\{\alpha g^x\}$, $x = 1, 2, \dots, P$, that fall in the interval $\mathfrak{M}$.

N. M. Korobov ([1]) constructed a number α such that, for any interval $\mathfrak{M}$ located in the interval $[0, 1)$, the estimate

$$N_P(\mathfrak{M}) = P \operatorname{mes} \mathfrak{M} + O(\sqrt{P}),$$

holds, where $\operatorname{mes} \mathfrak{M}$ is the length of the interval $\mathfrak{M}$.

A. G. Postnikov ([2]), using the method of trigonometric sums, constructed a number α such that, for any interval $\mathfrak{M}$ located in the interval $[0, 1)$, the estimate

$$N_P(\mathfrak{M}) = P \operatorname{mes} \mathfrak{M} + O\left(\frac{\sqrt{P}}{\sqrt[8]{\log P}} \log\log P\right)$$

holds.

In the present note we construct a number α such that, for any interval $\mathfrak{M}$ located in $[0, 1)$, the following estimate will hold:

$$N_P(\mathfrak{M}) = P \operatorname{mes} \mathfrak{M} + O\left(\frac{\sqrt{P}}{\sqrt[4]{\log P}} (\log\log P)^{3/2}\right).$$

We follow N. M. Korobov' s idea of constructing numbers α with the help of normal periodic systems ([3]). Denote by $\rho_n(g)$ a normal periodic system, and by $\rho'_n(g)$ the system obtained from $\rho_n(g)$ by discarding the last $n-1$ digits. Let

$$\varphi(j) = \left[\frac{g^j}{\sqrt{j}}\right], \qquad j = 1, 2, \dots.$$

Consider the number α defined in the following way:

$$\alpha = 0, \underbrace{\rho'_1(g)\ldots\rho'_1(g)}_{\varphi(1)\ \text{times}} \cdots \underbrace{\rho'_\mu(g)\ldots\rho'_\mu(g)}_{\varphi(\mu)\ \text{times}} \cdots \tag{1}$$

Theorem. Let the number α be defined by the string (1). For any interval $\mathfrak{M}$ located in $[0,1)$, the formula

$$N_P(\mathfrak{M}) = P\ \text{mes}\,\mathfrak{M} + O\left(\frac{\sqrt{P}}{\sqrt[4]{\log P}}(\log\log P)^{3/2}\right)$$

holds.

Proof. It suffices to prove the theorem for intervals $\mathfrak{M}$ of the form $(0,y)$, $0<y<1$. Denote by S_μ the number of g-ary digits of the number α up to the beginning of the first system $\rho'_\mu(g)$. Denote by $S_{\mu,\lambda}$ the number of g-ary digits in the number α up to the beginning of the λ-th system $\rho'_\mu(g)$ in order (obviously, $S_\mu = S_{\mu,1}$):

$$S_{\mu,\lambda} = \sum_{j=1}^{\mu-1}\varphi(j)g^j + (\lambda-1)g^\mu.$$

Arguing as in ([2]), we obtain

$$N_{S_{\mu,\lambda}}(\mathfrak{M}) = yS_{\mu,\lambda} + O\left(\frac{g^\mu}{\sqrt{\mu}}\right).$$

Let us note that

$$\mu = \frac{1}{2}\frac{\log P}{\log g} + \frac{1}{4}\frac{\ln g\ \log P}{\log g} + O(1).$$

Let

$$P = S_{\mu,\lambda} + P_1, \qquad 0 \le P_1 \le g^\mu - 1,$$

$$N_P(\mathfrak{M}) = YP + N_P(\mathfrak{M}) - N_{S_{\mu,\lambda}}(\mathfrak{M}) - YP_1 + O\left(\frac{g^\mu}{\sqrt{\mu}}\right).$$

Two cases may occur:

1) $0 \le P_1 \le \dfrac{g^\mu}{\sqrt{\mu}}$. Then

$$N_P(\mathfrak{M}) - N_{S_{\mu,\lambda}}(\mathfrak{M}) - YP_1 = O\left(\frac{g^\mu}{\sqrt{\mu}}\right).$$

2) In the case $g^\mu > P_1 \geq \dfrac{g^\mu}{\sqrt{\mu}}$ we shall rely on the following lemma.

Lemma. Let $\mathfrak{M}$ be any interval of the form $(0, Y)$, $0 < y < 1$. Let n, l be natural parameters, $4g^n \leq l \leq \dfrac{P_1 - 1}{n}$. $N_{P_1}(\mathfrak{M})$ denotes the number of hits of the fractional parts $\{\omega g^x\}$, $x = 1, 2, \dots, P_1$, in $\mathfrak{M}$. For sufficiently large P_1 the estimate

$$|N_P(\mathfrak{M}) - YP| < C\left(g^{nlnl}\sqrt{l}\, N_{P_1}^{(nl)} \exp\left[-\frac{l}{18g^{2n}}\right] + ln + \frac{P_1}{g^n}\right),$$

holds, where C is an absolute constant.

The proof of this lemma is carried out by the method proposed by I. I. Piatetski-Shapiro ([4]). We shall not give the proof of this lemma.

Take as the number ω the number

$$\omega = \{g^{S_{\mu,\lambda}}\alpha\}.$$

The g-adic expansion of the number ω has the form

$$\omega = \underbrace{\rho'_\mu(g) \dots \rho'_\mu(g)}_{\varphi(\mu)-(\lambda-1) \text{ times}} \quad \underbrace{\rho'_{\mu+1}(g) \dots \rho'_{\mu+1}(g)}_{\varphi(\mu+1) \text{ times}} \dots$$

Take

$$n = \left[\frac{\log\sqrt{\dfrac{\log P_1}{(\log\log P_1)^3}}}{\log g}\right], \qquad l = \left[\frac{\mu}{n}\right] + 1.$$

Applying the lemma, we obtain

$$\left|N_P(\mathfrak{M}) - N_{S_{\mu,\lambda}}(\mathfrak{M}) - YP_1\right| \leq$$

$$\leq C\left(g^\mu\left(\frac{\log P_1}{\log\log P_1}\right)^2 N_{P_1}^{(nl)} e^{-C_0(\log\log P_1)^2} + \frac{P_1(\log\log P_1)^{3/2}}{\sqrt{\log P_1}}\right) \leq$$

$$\leq Cg^{\mu}\left(\left(\frac{\log P_1}{\log\log P_1}\right)^2 N_{P_1}^{(nl)} e^{-C_0(\log\log P_1)^2} + \frac{(\log\log P_2)^{3/2}}{\sqrt{\log P_1}}\right).$$

Since $nl \geqslant \mu$, and the first g^{μ} g-ary digits of ω form the system $\hat{\rho}_{\mu}(g)$, it follows that $N_{P_1}^{(nl)} = 1$. For sufficiently large P_1,

$$\left(\frac{\log P_1}{\log\log P_1}\right)^2 e^{-C_0(\log\log P_1)^2} < \frac{(\log\log P_1)^{3/2}}{\sqrt{\log P_1}}.$$

Taking into account that $C_2\mu \leqslant \log P_1 \leqslant C_1\mu$, where C_1 and C_2 are positive constants, we obtain

$$N_P(\mathfrak{M}) - N_{S_{\mu},\lambda}(\mathfrak{M}) - YP_1 = O\left(\frac{g^{\mu}(\log\mu)^{3/2}}{\sqrt{\mu}}\right).$$

Thus, in both cases

$$N_P(\mathfrak{M}) - YP = O\left(\frac{g^{\mu}(\log\mu)^{3/2}}{\sqrt{\mu}}\right).$$

Since

$$\mu = \frac{1}{2}\frac{\log P}{\log g} + \frac{1}{4}\frac{\log\log P}{\log g} + O(1),$$

we have

$$N_P(\mathfrak{M}) = YP + O\left(\frac{\sqrt{P}}{\sqrt[4]{\log P}}(\log\log P)^{3/2}\right).$$

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Note: Figure translations are in progress. See original paper for figures.