Reports of the Academy of Sciences of the USSR

MATHEMATICAL PHYSICS

1962

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1962. Volume 144, No. 5

MATHEMATICAL PHYSICS

V. V. TOLMACHEV

THE PROBLEM OF TWO-PARTICLE BOUND STATES FOR A NONIDEAL FERMI-DIRAC SYSTEM

(Presented by Academician N. N. Bogolyubov on 29 I 1962)

In quantum field theory (^{1}), whose mathematical methods have recently been applied with such success to problems of quantum statistics, the two-particle bound state is studied, as is well known, with the aid of the Bethe–Salpeter equation.

The Bethe–Salpeter equation is based on a completely exact relation connecting the two-particle Green' s function, the one-particle Green' s function, and the so-called Schwinger interaction operator (the latter specifies the concrete form of the interaction, and it should be regarded as known).

This relation by itself is, of course, insufficient for a complete study of the two-particle bound state, since it contains two unknown functions. However, by making certain approximations with respect to the one-particle Green' s function entering the relation, one can always turn the indicated relation into an equation (this is what is usually done in quantum field theory (^{1})).

Since there are serious doubts, at least in the case of a nonideal Fermi-Dirac system*, as to the possibility of proceeding in this way (this, of course, excludes the possibility of investigating the influence of a two-particle bound state on the one-particle Green' s function), one should try to use some other additional and independent relation connecting the unknown two-particle and one-particle Green' s functions**.

Below we give the formulation of completely exact systems of equations for studying a two-particle state particle–particle or hole–hole, and for studying a two-particle state particle–hole in the case of a nonideal Fermi-Dirac system with a four-fermion interaction.

In what follows we shall restrict ourselves to considering the system at a temperature equal to zero. The system is characterized by the total Hamiltonian H, which is assumed to commute with the operator of the total number of particles of the system N (the law of conservation of the total number of particles). The system is assumed to be contained in a volume V.

The one-particle and two-particle Green' s functions are defined, as usual, as averages over the true ground state of the system of chronological products of field operators in the Heisenberg representation:

$$G(p_1t_1;p_1't_1') = \langle T(a_{p_1}(t_1)a^+_{p_1'}(t_1'))\rangle; \tag{1}$$

$$G(p_1t_1,p_2t_2;p_1't_1',p_2't_2') = \langle T(a_{p_1}(t_1)a_{p_2}(t_2)a^+_{p_2'}(t_2')a^+_{p_1'}(t_1'))\rangle, \tag{2}$$

where p denotes the wave vector and spin projection.

* Such a situation is encountered in the microscopic theory of superconductivity (^{2}).

** In the case of a nonideal Fermi-Dirac system with a four-fermion interaction, a relation is known (^{3}) that connects the one-particle Green' s function, the two-particle Green' s function, and the initial "bare" interaction, whereas the Bethe–Salpeter equation contains, as is known, the "dressed" interaction characterized by the Schwinger interaction operator. It should also be noted that this relation was used in addition to the Bethe–Salpeter equation (^{3,4}). The unsatisfactory nature of such an approach, however, is clear from the fact that two functions (for the "bare" and the "dressed" interaction) characterizing one and the same interaction cannot, in essence, enter the theory independently.

The relation underlying the Bethe–Salpeter equation discussed above has the following form:

$$G(p_1t_1,\ p_2t_2;\ p_1't_1',\ p_2't_2') =$$

$$= G(p_1t_1;\ p_1't_1')G(p_2t_2;\ p_2't_2') - G(p_1t_1;\ p_2't_2')G(p_2t_2;\ p_1't_1') -$$

$$-\frac{i}{\hbar}\frac{1}{V}\int_{-\infty}^{+\infty}dt_1'''\sum_{p_1'''}\int_{-\infty}^{+\infty}dt_2'''\sum_{p_2'''}\int_{-\infty}^{+\infty}dt_1''\sum_{p_1''}\int_{-\infty}^{+\infty}dt_2''\sum_{p_2''}G(p_1t_1;\ p_1'''t_1''')G(p_2t_2;\ p_2'''t_2''')\times$$

$$\times I(p_1'''t_1''',\ p_2'''t_2''';\ p_1''t_1'',\ p_2''t_2'')G(p_1''t_1'',\ p_2''t_2'';\ p_1't_1',\ p_2't_2'), \tag{3}$$

where I denotes the Schwinger interaction operator.

Relation (3) connects the one-particle Green function, the two-particle Green function, and the Schwinger interaction operator. Let us now proceed to derive one more independent relation connecting these quantities.

Take the definitions (1) and (2) of the Green functions. Put $p_2' = p_2$ and $t_2' = t_2 + 0$ in (2), and sum the right- and left-hand sides over p_2. Then, taking into account that the operator of the total number of particles

$$N = \sum_p a_p^+ a_p$$

commutes with the full Hamiltonian, and also that

$$N a_p = a_p (N - 1)$$

and

$$N a_p^+ = a_p^+ (N + 1)$$

by virtue of the anticommutation permutation relations for $a_p,\ a_p^+$, we obtain, using (1),

$$\sum_{p_2} G(p_1 t_1,\ p_2 t_2;\ p_1' t_1',\ p_2 t_2 + 0) = -N G(p_1 t_1;\ p_1' t_1') +$$

$$+ G(p_1 t_1;\ p_1' t_1')(-\theta(t_1 - t_2)\theta(t_2 - t_1') + \theta(t_1' - t_2)\theta(t_2 - t_1)). \qquad (4)$$

Combining (4) and (5), we obtain the desired additional relation connecting the one-particle Green function, the two-particle Green function, and the Schwinger interaction operator:

$$G(p_1 t_1;\ p_1' t_1')(-\theta(t_1 - t_2)\theta(t_2 - t_1') + \theta(t_1' - t_2)\theta(t_2 - t_1)) =$$

$$= -\sum_{p_2} G(p_1 t_1;\ p_2 t_2) G(p_2 t_2;\ p_1' t_1') -$$

$$-\frac{i}{\hbar}\frac{1}{V}\int_{-\infty}^{+\infty} dt_1''' \sum_{p_1'''} \int_{-\infty}^{+\infty} dt_2''' \sum_{p_2'''} \int_{-\infty}^{+\infty} dt_1'' \sum_{p_1''} \int_{-\infty}^{+\infty} dt_2'' \sum_{p_2''} \sum_{p_2} G(p_1 t_1;\ p_1''' t_1''') G(p_2 t_2;\ p_2''' t_2''') \times$$

$$\times I(p_1''' t_1''', p_2''' t_2''';\ p_1'' t_1'',\ p_2'' t_2'') G(p_1'' t_1'',\ p_2'' t_2'';\ p_1' t_1',\ p_2 t_2). \qquad (5)$$

Considering the Schwinger interaction operator to be a known function, both relations (3) and (5) may be understood as a system of two equations for the unknown two-particle and one-particle Green functions. Let us pass to the energy representation

$$G(p_1E_1;\ p_1'E_1') = \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_1' e^{\frac{i}{\hbar}E_1 t_1 - \frac{i}{\hbar}E_1' t_1'} G(p_1t_1;\ p_1't_1')$$

and so on, and use the spatial and temporal translational invariance of the problem, by virtue of which

$$G(p_1E_1;\ p_1'E_1') = 2\pi\hbar\delta(E_1 - E_1')\Delta(p_1 - p_1')G(p_1E_1),$$

$$G(p_1E_1, p_2E_2;\ p_1'E_1', p_2'E_2') = 2\pi\hbar\delta(E_1 + E_2 - E_1' - E_2')\Delta(p_1 + p_2 - p_1' - p_2')\times$$

$$\times\frac{1}{V}G((p_2 - p_1)/2,\ (E_2 - E_1)/2;\ (p_2' - p_1')/2,\ (E_2' - E_1')/2;\ p_1 + p_2,\ E_1 + E_2),$$

$$I(p_1E_1, p_2E_2;\ p_1'E_1', p_2'E_2') = 2\pi\hbar\delta(E_1 + E_2 - E_1' - E_2')\Delta(p_1 + p_2 - p_1' - p_2')\times$$

$$\times I((p_2 - p_1)/2,\ (E_2 - E_1)/2;\ (p_2' - p_1')/2,\ (E_2' - E_1')/2;\ p_1 + p_2,\ E_1 + E_2).$$

Then the system of equations (3) and (5) takes the form

$$\frac{i}{\hbar}\frac{\partial G(p_1E_1)}{\partial E_1} = G^2(p_1E_1) - G(p_1E_1)\frac{i}{\hbar}\frac{1}{V}\sum_p \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} d\mathcal{E}\, G(P - p_1, \mathcal{E} - E_1)\times$$

$$\times\frac{1}{V}\sum_{p''}\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dE''\ I(P/2 - p_1, \mathcal{E}/2 - E_1; p''E''; P\mathcal{E})\times$$

$$\times G(p''E''; P/2 - p_1, \mathcal{E}/2 - E_1; P\mathcal{E}); \tag{6}$$

$$G(pE; p'E'; P\mathcal{E}) = G(P/2 - p, \mathcal{E}/2 - E)\, G(P/2 + p, \mathcal{E}/2 + E)\, 2\pi\hbar V\times$$

$$\times\left(\delta(E - E')\Delta(p - p') - \delta(E + E')\Delta(p + p')\right) -$$

$$-\frac{i}{\hbar}\frac{1}{V}\sum_{p''}\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dE''\, G(P/2-p,\mathcal{E}/2-E)\, G(P/2+p,\mathcal{E}/2+E)\times$$

$$\times I(pE;p''E'';P\mathcal{E})\, G(p''E'';p'E';P\mathcal{E}). \tag{7}$$

Let us note the form, unusual for quantum field theory, of equation (6), in that it contains a derivative with respect to the energy variable.

The system of equations (6), (7) is convenient for studying, in a nonideal Fermi-Dirac system, a two-particle bound state which is a bound particle-particle or hole-hole state. In a similar way one can write a system of equations for studying a particle-hole state. Essential here, once again, is the use of relation (4). In addition, instead of the Schwinger interaction operator I one should use another interaction operator J^*.

Analogously to (3) and (5), we have

$$G(p_1t_1,p_2t_2;p_1't_1',p_2't_2')=$$

$$=G(p_1t_1;p_1't_1')G(p_2t_2;p_2't_2')-G(p_1t_1;p_2't_2')G(p_2t_2;p_1't_1')+$$

$$+\frac{i}{\hbar}\frac{1}{V}\int_{-\infty}^{+\infty} dt_1''\sum_{p_1''}\int_{-\infty}^{+\infty} dt_2''\sum_{p_2''}\int_{-\infty}^{+\infty} dt_1'''\sum_{p_1'''}\int_{-\infty}^{+\infty} dt_2'''\sum_{p_2'''}G(p_1t_1;p_1''t_1'')G(p_1'''t_1''';p_1't_1')\times$$

$$\times J(p_1''t_1'',p_2'''t_2''',p_1'''t_1''',p_2''t_2'')\,(G(p_2''t_2'',p_2t_2;p_2'''t_2''',p_2't_2')-$$

$$-G(p_2''t_2'';p_2'''t_2''')G(p_2t_2;p_2't_2')) \tag{8}$$

$$G(p_1t_1;p_1't_1')\,(-\theta(t_1-t_2)\theta(t_2-t_1')+\theta(t_1'-t_2)\theta(t_2-t_1))=$$

$$=-\sum_{p_2}G(p_1t_1;p_2t_2)G(p_2t_2;p_1't_1')+$$

$$+\frac{i}{\hbar}\frac{1}{V}\int_{-\infty}^{+\infty} dt_1''\sum_{p_1''}\int_{-\infty}^{+\infty} dt_2''\sum_{p_2''}\int_{-\infty}^{+\infty} dt_1'''\sum_{p_1'''}\int_{-\infty}^{+\infty} dt_2'''\sum_{p_2'''}\sum_{p_2}G(p_1t_1;p_1''t_1'')G(p_1'''t_1''';p_1't_1')\times$$

$$\times J(p_1''t_1'', p_2'''t_2'''; p_1'''t_1''', p_2''t_2'')G(p_2''t_2''; p_2'''t_2''')\times$$

$$\times\,(-\theta(t_2''-t_2)\theta(t_2-t_2''')+\theta(t_2'''-t_2)\theta(t_2-t_2''))\,. \tag{9}$$

* The interaction operators $I(12,1'2')$ and $J(12,1'2')$ in the Feynman diagram technique (5) are represented by sums of all possible diagrams with two incoming and two outgoing ends (joined to the four vertices and not belonging to the diagram itself). These diagrams: 1) are topologically connected and contain no vacuum loops; 2) from all four of their ends it is impossible to "pull out" any proper energy part; 3) it is impossible to represent them in the form of two parts connected to each other only by two electron lines, and in the case of I, 1 and 2 are in one part and $1'$ and $2'$ in the other, whereas in the case of J, 1 and $1'$ are in one part and 2 and $2'$ in the other; 4) in the case of I, in diagrams, starting from $1'$ (or $2'$) along electron lines, one can arrive at 1 (or 2).

Let us pass to the energy representation and use the spatial and temporal translational invariance of the problem

$$G(p_1E_1;\ p_1'E_1')=2\pi\hbar\delta(E_1-E_1')\Delta(p_1-p_1')G(p_1E_1),$$

$$G(p_1E_1,\ p_2E_2;\ p_1'E_1',\ p_2'E_2')=2\pi\hbar\delta(E_1+E_2-E_1'-E_2')\Delta(p_1+p_2-p_1'-p_2')\times$$
$$\times\frac{1}{V}G((p_1+p_1')/2,\ (E_1+E_1')/2;\ (p_2'+p_2)/2,\ (E_2'+E_2)/2;\ p_1-p_1',\ E_1-E_1'$$

$$J(p_1E_1,\ p_2E_2;\ p_1'E_1',\ p_2'E_2')=2\pi\hbar\delta(E_1+E_2-E_1'-E_2')\Delta(p_1+p_2-p_1'-p_2')\times$$
$$\times J((p_1+p_1')/2,\ (E_1+E_1')/2;\ (p_2'+p_2)/2,\ (E_2'+E_2)/2;\ p_1-p_1',\ E_1-E_1').$$

Then equations (8) and (9) are transformed into the following form

$$\frac{\hbar}{i}\frac{\partial}{\partial E_1}G(p_1E_1)=G^2(p_1E_1)+G^2(p_1E_1)\times$$
$$\times\frac{1}{V}\sum_{p_2'}\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}dE_2'\,J(p_1E_1;\ p_2'E_2';\ 0,\ 0)\,\frac{\partial}{\partial E_2'}G(p_2'E_2'); \tag{10}$$

$$\begin{aligned} G(pE,\ p'E';\ P\mathcal{E}) = {} & G(P/2+p,\ \mathcal{E}/2+E)\, G(-P/2+p',\ -\mathcal{E}/2+E')\, 2\pi\hbar V \times \\ & \times (\delta(\mathcal{E})\Delta(P) - \delta(E-E')\Delta(p-p')) + \\ & + G(P/2+p,\ \mathcal{E}/2+E)\, G(-P/2+p,\ -\mathcal{E}/2+E) \times \\ & \times \frac{i}{\hbar}\frac{1}{V}\sum_{p''}\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dE''\, J(pE;\ p''E'';\ P\mathcal{E}) G(p''E'';\ p'E';\ P\mathcal{E}) - \\ & - G(P/2+p,\ \mathcal{E}/2+E)\, G(-P/2+p,\ -\mathcal{E}/2+E) \times \\ & \times G(-P/2+p',\ -\mathcal{E}/2+E')\, 2\pi\hbar\delta(\mathcal{E}) V \Delta(P) \times \\ & \times \frac{i}{\hbar}\frac{1}{V}\sum_{p''_2}\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dE''_2\, J(pE;\ p''_2 E''_2;\ P\mathcal{E}) G(p''_2 E''_2). \end{aligned} \tag{11}$$

Let us note an interesting feature of equations (10), (11). In essence they do not constitute a system of two coupled equations, since the two-particle Green' s function has dropped out of equation (10).

The system of equations (6), (7), as it seems to the author, forms the basis for the existing microscopic theory of superconductivity and the theory of nuclear matter of Brueckner. The system of equations (10), (11) forms the basis for the existing plasma theory.

The author expresses his gratitude to Academician N. N. Bogolyubov for a useful discussion of the work.

Physico-Chemical Institute
named after L. Ya. Karpov

Received
29 I 1962

REFERENCES CITED

[1] J. Schwinger, Proc. Am. Acad. Arts and Sci., **37**, 452 (1951); H. A. Bethe, E. E. Salpeter, Phys. Rev., **84**, 1232 (1951); M. Gell-Mann, F. Low, Phys. Rev., **84**, 350 (1951); G. C. Wick, Phys. Rev., **96**, 1124 (1954); R. E. Cutkosky, Phys. Rev., **96**, 1135 (1954); S. Okubo, D. Feldman, Phys. Rev., **117**, 279, 292 (1960).

[2] J. Bardeen, L. N. Cooper, J. R. Schrieffer, Phys. Rev., **108**, 1175 (1957); N. N. Bogolyubov, V. V. Tolmachev, D. V. Shirkov, *A New Method in the Theory of Superconductivity*, USSR Academy of Sciences Press, 1958.

[3] A. Klein, R. Prange, Phys. Rev., **112**, 994, 1008 (1958).

[4] W. Wild, Z. f. Phys., **158**, 322 (1960).

[5] J. Goldstone, Proc. Roy. Soc., **239**, 267 (1957).